RAY'S

MATHEMATICAL

SERIES.

ECLECTIC EDUCATIONAL SERIES.

TREATISE

ON

PLANE AND SOLID GEOMETRY:

FOR

COLLEGES, SCHOOLS, AND PRIVATE STUDENTS.

WRITTEN FOR THE MATHEMATICAL COURSE OF

JOSEPH RAY, M.D.,

BY

ELI T. TAPPAN, M.A.,

PROFESSOR OF MATHEMATICS, MT. AUBURN INSTITUTE.

VAN ANTWERP, BRAGG & CO.,

137 WALNUT STREET, CINCINNATI.

28 BOND STREET, NEW YORK.

RAY'S SERIES,

EMBRACING

A Thorough and Progressive Course in Arithmetic, Algebra, and the Higher Mathematics.

Primary Arithmetic.
Intellectual Arithmetic.
Rudiments of Arithmetic.
Practical Arithmetic.
Higher Arithmetic.
Test Examples in Arithmetic.
New Elementary Algebra.
New Higher Algebra.

Plane and Solid Geometry. By ELI T. TAPPAN, A.M., *Pres't Kenyon College.* 12*mo, cloth,* 276 *pp.*

Geometry and Trigonometry. By ELI T. TAPPAN, A.M. *Pres't Kenyon College.* 8*vo, sheep,* 420 *pp.*

Analytic Geometry. By GEO. H. HOWISON, A.M., *Prof. in Mass. Institute of Technology.* Treatise on Analytic Geometry, especially as applied to the Properties of Conics: including the Modern Methods of Abridged Notation.

Elements of Astronomy. By S. H. PEABODY, A.M., *Prof. of Physics and Civil Engineering, Amherst College.* Handsomely and profusely illustrated. 8*vo, sheep,* 336 *pp.*

KEYS.

Ray's Arithmetical Key (*To Intellectual and Practical*);
Key to Ray's Higher Arithmetic;
Key to Ray's New Elementary and Higher Algebras.

The Publishers furnish Descriptive Circulars of the above Mathematical Text-Books, with Prices and other information concerning them.

PREFACE.

The science of Elementary Geometry, after remaining nearly stationary for two thousand years, has, for a century past, been making decided progress. This is owing, mainly, to two causes: discoveries in the higher mathematics have thrown new light upon the elements of the science; and the demands of schools, in all enlightened nations, have called out many works by able mathematicians and skillful teachers.

Professor Hayward, of Harvard University, as early as 1825, defined parallel lines as lines having the same direction. Euclid's definitions of a straight line, of an angle, and of a plane, were based on the idea of direction, which is, indeed, the essence of form. This thought, employed in all these leading definitions, adds clearness to the science and simplicity to the study. In the present work, it is sought to combine these ideas with the best methods and latest discoveries in the science.

By careful arrangement of topics, the theory of each class of figures is given in uninterrupted connection. No attempt is made to exclude any method of demonstration, but rather to present examples of all.

The books most freely used are, "Cours de géométrie élémentaire, par A. J. H. Vincent et M. Bourdon;" "Géométrie théorique et pratique, etc., par H. Sonnet;" "Die reine elementar-mathematik, von Dr. Martin Ohm;" and "Treatise on Geometry and its application to the Arts, by Rev. D. Lardner."

The subject is divided into chapters, and the articles are numbered continuously through the entire work. The convenience of this arrangement for purposes of reference, has caused it to be adopted by a large majority of writers upon Geometry, as it had been by writers on other scientific subjects.

In the chapters on Trigonometry, this science is treated as a branch of Algebra applied to Geometry, and the trigonometrical functions are defined as ratios. This method has the advantages of being more simple and more brief, yet more comprehensive, than the ancient geometrical method.

For many things in these chapters, credit is due to the works of Mr. I. Todhunter, M. A., St. John's College, Cambridge.

The tables of logarithms of numbers and of sines and tangents have been carefully read with the corrected edition of Callet, with the tables of Dr. Schrön, and with those of Babbage.

ELI T. TAPPAN.

Ohio University, *Jan.* 1, 1868.

CONTENTS.

PART FIRST.—INTRODUCTORY.

CHAPTER I.

PRELIMINARY.

CHAPTER II.

THE SUBJECT STATED.

PART SECOND.—PLANE GEOMETRY.

CHAPTER III.

STRAIGHT LINES.

CHAPTER VIII.

CIRCLES.

PART THIRD.—GEOMETRY OF SPACE.

CHAPTER IX.

STRAIGHT LINES AND PLANES.

CHAPTER X.

POLYEDRONS.

CHAPTER XI.

SOLIDS OF REVOLUTION.

PART FOURTH.—TRIGONOMETRY.

CHAPTER XII.

PLANE TRIGONOMETRY.

CHAPTER XIII.

SPHERICAL TRIGONOMETRY.

CHAPTER XIV.

LOGARITHMS.

TABLES.

ELEMENTS

OF

GEOMETRY.

CHAPTER I.—PRELIMINARY.

Article 1. Before the student begins the study of geometry, he should know certain principles and definitions, which are of frequent use, though they are not peculiar to this science. They are very briefly presented in this chapter.

LOGICAL TERMS.

2. Every statement of a principle is called a Proposition.

Every proposition contains the subject of which the assertion is made, and the property or circumstance asserted.

When the subject has some condition attached to it, the proposition is said to be conditional.

The subject, with its condition, if it have any, is the Hypothesis of the proposition, and the thing asserted is the Conclusion.

Each of two propositions is the Converse of the other, when the two are such that the hypothesis of either is the conclusion of the other.

3. A proposition is either *theoretical,* that is, it declares that a certain property belongs to a certain thing; or it is *practical,* that is, it declares that something can be done.

Propositions are either *demonstrable,* that is, they may be established by the aid of reason; or they are *indemonstrable,* that is, so simple and evident that they can not be made more so by any course of reasoning.

A THEOREM is a demonstrable, theoretical proposition.

A PROBLEM is a demonstrable, practical proposition.

An AXIOM is an indemonstrable, theoretical proposition.

A POSTULATE is an indemonstrable, practical proposition.

A proposition which flows, without additional reasoning, from previous principles, is called a COROLLARY. This term is also frequently applied to propositions, the demonstration of which is very brief and simple.

4. The reasoning by which a proposition is proved is called the DEMONSTRATION.

The explanation how a thing is done constitutes the SOLUTION of a problem.

A DIRECT DEMONSTRATION proceeds from the premises by a regular deduction.

An INDIRECT DEMONSTRATION attains its object by showing that any other hypothesis or supposition than the one advanced would involve a contradiction, or lead to an impossible conclusion. Such a conclusion may be called absurd, and hence the Latin name of this method of reasoning—*reductio ad absurdum.*

A work on Geometry consists of definitions, propositions, demonstrations, and solutions, with introductory or explanatory remarks. Such remarks sometimes have the name of scholia.

5. Remark.—The student should learn each proposition, so as to state separately the hypothesis and the conclusion, also the condition, if any. He should also learn, at each demonstration, whether it is direct or indirect; and if indirect, then what is the false hypothesis and what is the absurd conclusion. It is a good exercise to state the converse of a proposition.

In this work the propositions are first enounced in general terms. This general enunciation is usually followed by a particular statement of the principle, as a fact, referring to a diagram. Then follows the demonstration or solution. In the latter part of the work these steps are frequently shortened.

The student is advised to conclude every demonstration with the general proposition which he has proved.

The student meeting a reference, should be certain that he can state and apply the principle referred to.

GENERAL AXIOMS.

6. *Quantities which are each equal to the same quantity, are equal to each other.*

7. *If the same operation be performed upon equal quantities, the results will be equal.*

For example, if the same quantity be separately added to two equal quantities, the sums will be equal.

8. *If the same operation be performed upon unequal quantities, the results will be unequal.*

Thus, if the same quantity be subtracted from two unequal quantities, the remainder of the greater will exceed the remainder of the less.

9. *The whole is equal to the sum of all the parts.*

10. *The whole is greater than a part.*

EXERCISE.

11. What is the hypothesis of the first axiom? *Ans.* If several quantities are each equal to the same quantity.

What is the subject of the first axiom? *Ans.* Several quantities.

What is the condition of the first axiom? *Ans.* That they are each equal to the same quantity.

What is the conclusion of the first axiom? *Ans.* Such quantities are equal to each other.

Give an example of this axiom.

RATIO AND PROPORTION

12. All mathematical investigations are conducted by comparing quantities, for we can form no conception of any quantity except by comparison.

13. In the comparison of one quantity with another, the relation may be noted in two ways: either, first, how much one exceeds the other; or, second, how many times one contains the other.

The result of the first method is the difference between the two quantities; the result of the second is the RATIO of one to the other.

Every ratio, as it expresses "how many times" one quantity contains another, is a number. That a ratio and a number are quantities of the same kind, is further shown by comparing them; for we can find their sum, their difference, or the ratio of one to the other.

When the division can be exactly performed, the ratio is a whole number; but it may be a fraction, or a radical, or some other number incommensurable with unity.

14. The symbols of the quantities from whose comparison a ratio is derived, are frequently retained in its expression. Thus,

The ratio of a quantity represented by a to another represented by b, may be written $\frac{a}{b}$.

A ratio is usually written $a : b$, and is read, a is to b.

This retaining of the symbols is merely for convenience, and to show the derivation of the ratio; for a ratio may be expressed by a single figure, or by any other symbol, as 2, m, $\sqrt{3}$, or π. But since every ratio is a number, therefore, when a ratio is thus expressed by means of two terms, they must be understood to represent two numbers having the same relation as the given quantities.

The second term is the standard or unit with which the first is compared.

So, when the ratio is expressed in the form of a fraction, the first term, or ANTECEDENT, becomes the numerator, and the second, or CONSEQUENT, the denominator.

15. A PROPORTION is the equality of two ratios, and is generally written,

$$a : b :: c : d,$$

and is read, a is to b as c is to d,

but it is sometimes written,

$$a : b = c : d,$$

or it may be, $\frac{a}{b} = \frac{c}{d}$,

all of which express the same thing: that a contains b exactly as often as c contains d.

The first and last terms are the EXTREMES, and the second and third are the MEANS of a proportion.

The fourth term is called the FOURTH PROPORTIONAL of the other three.

A series of equal ratios is written,

$$a : b :: c : d :: e : f, \text{ etc.}$$

When a series of quantities is such that the ratio of each to the next following is the same, they are written,

$$a : b : c : d, \text{ etc.}$$

Here, each term, except the first and last, is both antecedent and consequent. When such a series consists of three terms, the second is the MEAN PROPORTIONAL of the other two.

16. Proposition.—*The product of the extremes of any proportion is equal to the product of the means.*

For any proportion, as

$$a : b :: c : d,$$

is the equation of two fractions, and may be written,

$$\frac{a}{b}=\frac{c}{d}.$$

Multiplying these equals by the product of the denominators, we have (7)

$$a\times d=b\times c,$$

or the product of the extremes equal to the product of the means.

17. Corollary.—The square of a mean proportional is equal to the product of the extremes. A mean proportional of two quantities is the square root of their product.

18. Proposition.—*When the product of two quantities is equal to the product of two others, either two may be the extremes and the other two the means of a proportion.*

Let $a\times d=b\times c$ represent the equal products.

If we divide by b and d, we have

$$\frac{a}{b}=\frac{c}{d}; \text{ or, } a : b :: c : d. \qquad \text{(1st.)}$$

If we divide by c and d, we have

$$\frac{a}{c}=\frac{b}{d}; \text{ or, } a : c :: b : d. \qquad \text{(2d.)}$$

If we arrange the equal products thus:

$$b\times c=a\times d,$$

and then divide by a and c, we have

$$b : a :: d : c. \qquad (3d.)$$

By similar divisions, the student may produce five other arrangements of the same quantities in proportion.

19. Proposition.—*The order of the terms may be changed without destroying the proportion, so long as the extremes remain extremes, or both become means.*

Let $a : b :: c : d$ represent the given proportion.

Then (16), we have $a \times d = b \times c$. Therefore (18), a and d may be taken as either the extremes or the means of a new proportion.

20. When we say the first term is to the third as the second is to the fourth, the proportion is taken by *alternation*, as in the second case, Article 18.

When we say the second term is to the first as the fourth is to the third, the proportion is taken *inversely*, as in the third case.

21. Proposition.—*Ratios which are equal to the same ratio are equal to each other.*

This is a case of the first axiom (6).

22. Proposition.—*If two quantities have the same multiplier, the multiples will have the same ratio as the given quantities.*

Let a and b represent any two quantities, and m any multiplier. Then the identical equation,

$$m \times a \times b = m \times b \times a,$$

gives the proportion,

$$m \times a : m \times b :: a : b \text{ (18)}.$$

23. Proposition.—*In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Let $a : b :: c : d :: e : f :: g : h$, etc., represent the equal ratios.

Therefore (16),

$$a \times d = b \times c$$
$$a \times f = b \times e$$
$$a \times h = b \times g$$

To these add $a \times b = b \times a$

$$a \times (b + d + f + h) = b \times (a + c + e + g).$$

Therefore (18),

$$a + c + e + g : b + d + f + h :: a : b.$$

This is called proportion by COMPOSITION.

24. Proposition.—*The difference between the first and second terms of a proportion is to the second, as the difference between the third and fourth is to the fourth.*

The given proportion,

$$a : b :: c : d,$$

may be written,

$$\frac{a}{b} = \frac{c}{d}.$$

Subtract the identical equation,

$$\frac{b}{b} = \frac{d}{d}.$$

The remaining equation,

$$\frac{a-b}{b} = \frac{c-d}{d},$$

may be written, $a - b : b :: c - d : d.$

This is called proportion by DIVISION.

25. Proposition.—*If four quantities are in proportion, their same powers are in proportion, also their same roots.*

Thus, if we have $a : b :: c : d,$

then, $a^2 : b^2 :: c^2 : d^2;$

also, $\sqrt{a} : \sqrt{b} :: \sqrt{c} : \sqrt{d}.$

These principles are corollaries of the second general axiom (7), since a proportion is an equation.

CHAPTER II.

THE SUBJECT STATED.

26. We know that every material object occupies a portion of space, and has extent and form.

For example, this book occupies a certain space; it has a definite extent, and an exact form. These properties may be considered separate, or abstract from all others. If the book be removed, the space which it had occupied remains, and has these properties, extent and form, and none other.

27. Such a limited portion of space is called a solid.

Be careful to distinguish the geometrical solid, which is a portion of space, from the solid body which occupies space.

Solids may be of all the varieties of extent and form that are found in nature or art, or that can be imagined.

28. The limit or boundary which separates a solid from the surrounding space is a surface. A surface is like a solid in having only these two properties, extent and form; but a surface differs from a solid in having no thickness or depth, so that a solid has one kind of extent which a surface has not.

As solids and surfaces have an abstract existence, without material bodies, so two solids may occupy the same space, entirely or partially. For example, the position which has been occupied by a book, may be now occupied by a block of wood. The solids represented by the book and block may occupy at once, to some extent, the same space. Their surfaces may meet or cut each other.

29. The limits or boundaries of a surface are lines. The intersection of two surfaces, being the limit of the parts into which each divides the other, is a line.

A line has these two properties only, extent and form; but a surface has one kind of extent which a line has not: a line differs from a surface in the same way that a surface does from a solid. A line has neither thickness nor breadth.

30. The ends or limits of a line are points. The intersections of lines are also points. A point is unlike either lines, surfaces, or solids, in this, that it has neither extent nor form.

31. As one line may be met by any number of others, and a surface cut by any number of others; so a line may have any number of points, and a surface any number of lines and points. And a solid may have any number of intersecting surfaces, with their lines and points.

DEFINITIONS.

32. These considerations have led to the following definitions:

A Point has only position, without extent.

A Line has length, without breadth or thickness.

A Surface has length and breadth, without thickness.

A Solid has length, breadth, and thickness.

33. A line may be measured only in one way, or, it may be said a line has only one dimension. A surface has two, and a solid has three dimensions. We can not conceive of any thing of more than three dimensions. Therefore, every thing which has extent and form belongs to one of these three classes.

The extent of a line is called its LENGTH; of a surface, its AREA; and of a solid, its VOLUME.

34. Whatever has only extent and form is called a MAGNITUDE.

GEOMETRY is the science of magnitude.

Geometry is used whenever the size, shape, or position of any thing is investigated. It establishes the principles upon which all measurements are made. It is the basis of Surveying, Navigation, and Astronomy.

In addition to these uses of Geometry, the study is cultivated for the purpose of training the student's powers of language, in the use of precise terms; his reason, in the various analyses and demonstrations; and his inventive faculty, in the making of new solutions and demonstrations.

THE POSTULATES.

35. Magnitudes may have any extent. We may conceive lines, surfaces, or solids, which do not extend beyond the limits of the smallest spot which represents a point; or, we may conceive them of such extent as to reach across the universe. The astronomer knows that his lines reach to the stars, and his planes extend beyond the sun. These ideas are expressed in the following

Postulate of Extent.—*A magnitude may be made to have any extent whatever.*

36. Magnitudes may, in our minds, have any form, from the most simple, such as a straight line, to that of the most complicated piece of machinery. We may conceive of surfaces without solids, and of lines without surfaces.

It is a useful exercise to imagine lines of various forms, extending not only along the paper or blackboard, but across the room. In the same way, surfaces and solids may be conceived of all possible forms.

The form of a magnitude consists in the relative position of the parts, that is, in the relative directions of the points. Every change of form consists in changing the relative directions of the points of the figure.

Every geometrical conception, however simple or complex, is composed of only two kinds of elementary thoughts—directions and distances. The directions determine its form, and the distances its extent.

Postulate of Form.—*The points of a magnitude may be made to have from each other any directions whatever, thus giving the magnitude any conceivable form.*

These two are all the postulates of geometry. They rest in the very ideas of space, form, and magnitude.

37. Magnitudes which have the same form while they differ in extent, are called SIMILAR.

Any point, line, or surface in a figure, and the similarly situated point, line, or surface in a similar figure, are called HOMOLOGOUS.

Magnitudes which have the same extent, while they differ in form, are called EQUIVALENT.

MOTION AND SUPERPOSITION.

38. The postulates are of constant use in geometrical reasoning.

Since the parts of a magnitude may have any position, they may change position. By this idea of motion, the mutual derivation of points, lines, surfaces, and solids may be explained.

The path of a point is a line, the path of a line may be a surface, and the path of a surface may be a solid. The time or rate of motion is not a subject of geometry, but the path of any thing is itself a magnitude.

39. By the idea of motion, one magnitude may be mentally applied to another, and their form and extent compared.

This is called the method of superposition, and is the most simple and useful of all the methods of demonstration used in geometry. The student will meet with many examples.

EQUALITY.

40. When two *equal* magnitudes are compared, it is found that they may coincide; that is, each contains the other. Since they coincide, every part of one will have its corresponding equal and coinciding part in the other, and the parts are arranged the same in both.

Conversely, if two magnitudes are composed of parts respectively equal and similarly arranged, one may be applied to the other, part by part, till the wholes coincide, showing the two magnitudes to be equal.

Each of the above convertible propositions has been stated as an axiom, but they appear rather to constitute the definition of equality.

FIGURES.

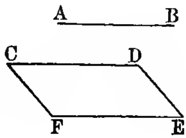
41. Any magnitude or combination of magnitudes which can be accurately described, is called a geometrical FIGURE.

Figures are represented by diagrams or drawings, and such representations are, in common language, called figures. A small spot is commonly called a point, and a long mark a line. But these have not only extent and form, but also color, weight, and other properties; and, therefore, they are not *geometrical* points and lines.

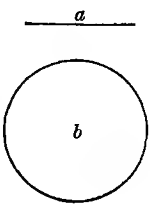
It is the more important to remember this distinction, since the point and line made with chalk or ink are constantly used to represent to the eye true mathematical points and lines.

42. The figure which is the subject of a proposition, together with all its parts, is said to be GIVEN. The additions to the figure made for the purpose of demonstration or solution, constitute the CONSTRUCTION.

43. In the diagrams in this work, points are designated by capital letters. Thus, the points A and B are at the extremities of the line.



Figures are usually designated by naming some of their points, as the line AB, and the figure CDEF, or simply the figure DF.



When it is more convenient to designate a figure by a single letter, the small letters are used. Thus, the line *a*, or the figure *b*.

LINES.

44. A STRAIGHT LINE is one which has the same direction throughout its whole extent.

A straight line may be regarded as the path of a point moving in one direction, turning neither up nor down, to the right or left.

45. A CURVED LINE is one which constantly changes its direction. The word *curve* is used for a *curved line.*

46. A line composed of straight lines, is called BROKEN. A line may be composed of curves, or of both curved and straight parts.

THE STRAIGHT LINE.

47. Problem.—*A straight line may be made to pass through any two points.*

48. Problem.—*There may be a straight line from any point, in any direction, and of any extent.*

These two propositions are corollaries of the postulates.

49. From a point, straight lines may extend in all directions. But we can not conceive that two separate straight lines can have the same direction from a common point. This impossibility is expressed by the following

Axiom of Direction.—*In one direction from a point, there can be only one straight line.*

50. Corollary.—From one point to another, there can be only one straight line

51. Theorem.—*If a straight line have two of its points common with another straight line, the two lines must coincide throughout their mutual extent.*

For, if they could separate, there would be from the point of separation two straight lines having the same direction, which is impossible (49).

52. Corollary.—Two fixed points, or one point and a certain direction, determine the position of a straight line.

53. If a straight line were turned upon two of its points as fixed pivots, no part of the line would change place. So any figure may revolve about a straight line, while the position of the line remains unchanged.

This property is peculiar to the straight line. If the curve BC were to revolve upon the two points B and C as pivots, then the straight line connecting these points would remain at rest, and the curve would revolve about it.

A straight line about which any thing revolves, is called its AXIS.

54. Axiom of Distance.—*The straight line is the shortest which can join two points.*

Therefore, the distance from one point to another is reckoned along a straight line.

55. There have now been given two postulates and two axioms. The science of geometry rests upon these four simple truths.

The possibility of every figure defined, and the truth of every problem, depend upon the postulates.

Upon the postulates, with the axioms, is built the demonstration of every principle.

SURFACES.

56. Surfaces, like lines, are classified according to their uniformity or change of direction.

A PLANE is a surface which never varies in direction.

A CURVED SURFACE is one in which there is a change of direction at every point.

THE PLANE.

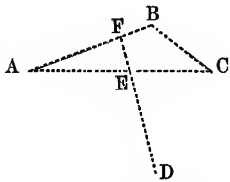
57. The plane surface and the straight line have the same essential character, sameness of direction. The plane is straight in every direction that it has.

A straight line and a plane, unless the extent be specified, are always understood to be of indefinite extent.

58. Theorem.—*A straight line which has two points in a plane, lies wholly in it, so far as they both extend.*

For if the line and surface could separate, one or the other would change direction, which by their definitions is impossible.

59. Theorem.—*Two planes having three points common, and not in the same straight line, coincide so far as they both extend.*



Let A, B, and C be three points which are not in one straight line, and let these points be common to two planes, which may be designated by the letters m and p. Let a straight line pass through the points A and B, a second through B and C, and a third through A and C.

Each of these lines (58) lies wholly in each of the planes m and p. Now it is to be proved that any point D, in the plane m, must also be in the plane p.

Let a line extend from D to some point of the line AC, as E. The points D and E being in the plane m, the whole line DE must be in that plane; and, therefore, if produced across the inclosed surface ABC, it will meet one of the other lines AB, BC, which also lie in that plane, say, at the point F. But the points F and E are both in the plane *p*. Therefore, the whole line FD, including the point D, is in the plane *p*.

In the same manner, it may be shown that any point which is in one plane, is also in the other, and therefore the two planes coincide.

60. Corollary.—Three points not in a straight line, or a straight line and a point out of it, fix the position of a plane.

61. Corollary.—That part of a plane on one side of any straight line in it, may revolve about the line till it meets the other part, when the two will coincide (53).

EXERCISES.

62. When a mechanic wishes to know whether a line is straight, he may apply another line to it, and observe if they coincide.

In order to try if a surface is plane, he applies a straight rule to it in many directions, observing whether the two touch throughout.

The mason, in order to obtain a plain surface to his marble, applies another surface to it, and the two are ground together until all unevenness is smoothed away, and the two touch throughout.

What geometrical principle is used in each of these operations?

In a diagram two letters suffice to mark a straight line. Why?

But it may require three letters to designate a curve. Why?

DIVISION OF SUBJECT.

63. By combinations of lines upon a plane, PLANE FIGURES are formed, which may or may not inclose an area.

By combinations of lines and surfaces, figures are formed in space, which may or may not inclose a volume.

In an elementary work, only a few of the infinite variety of geometrical figures that exist, are mentioned, and only the leading principles concerning those few.

Elementary Geometry is divided into PLANE GEOMETRY, which treats of plane figures, and GEOMETRY IN SPACE, which treats of figures whose points are not all in one plane.

In Plane Geometry, we will first consider lines without reference to area, and afterward inclosed figures.

In Geometry in Space, we will first consider lines and surfaces which do not inclose a space; and afterward, the properties of certain solids.

PLANE GEOMETRY.

CHAPTER III.

STRAIGHT LINES.

64. Problem.—*Straight lines may be added together, and one straight line may be subtracted from another.*

For a straight line may be produced to any extent. Therefore, the length of a straight line may be increased by the length of another line, or two lines may be added together, or we may find the sum of several lines (35).

Again, any straight line may be applied to another, and the two will coincide to their mutual extent. One line may be subtracted from another, by applying the less to the greater and noting the difference.

65. Problem.—*A straight line may be multiplied by any number.*

For several equal lines may be added together.

66. Problem.—*A straight line may be divided by another.*

By repeating the process of subtraction.

67. Problem.—*A straight line may be decreased in any ratio, or it may be divided into several equal parts.*

This is a corollary of the postulate of extent (35).

PROBLEMS IN DRAWING.

68. Exercises in linear drawing afford the best applications of the principles of geometry. Certain lines or combinations of lines being given, it is required to construct other lines which shall have certain geometrical relations to the former.

Except the paper and pencil, or blackboard and crayon, the only instruments used are the ruler and compasses; and all the required lines must be drawn by the aid of these only. The reason for this rule will be shown in the following chapter.

The ruler must have one edge straight. The compasses have two legs with pointed ends, which meet when the instrument is shut. For blackboard work, a stretched cord may be substituted for the compasses.

69. With the *ruler*, a straight line may be drawn on any plane surface, by placing the ruler on the surface and drawing the pencil along the straight edge.

A straight line may be drawn through any two points, after placing the straight edge in contact with the points.

A terminated straight line may be produced after applying the straight edge to a part of it, in order to fix the direction.

70. With the *compasses*, the length of a given line may be taken by opening the legs till the fine points are one on each end of the line. Then this length may be measured on the greater line as often as it will contain the less. A line may thus be produced any required length.

71. The student must distinguish between the problems of geometry and problems in drawing. The former state what can be done with pure geometrical magnitudes, and their truth depends upon showing that they are not incompatible with the nature of the given figure; for a geometrical figure can have any conceivable form or extent.

The problems in drawing corresponding to those above given, except the last, "to divide a given straight line into proportional or equal parts," are solved by the methods just described.

72. The complete discussion of a problem in drawing includes, besides the demonstration and solution, the showing whether the problem has only one solution or several, and the conditions of each.

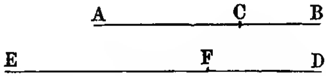
STRAIGHT LINES SIMILAR.

73. Theorem.—*Any two straight lines are similar figures.*

For each has one invariable direction. Hence, two straight lines have the same form, and can differ from each other only in their extent (37).

74. Any straight line may be diminished in any ratio (67), and may therefore be divided in any ratio.

The points in two lines which divide them in the same ratio are homologous points, by the definition (37). Thus, if the lines AB and ED are divided at the points C and F, so that AC : CB :: EF : FD,



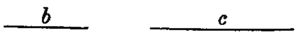
then C and F are homologous, or similarly situated points in these lines; AC and EF are homologous parts, and CB and FD are homologous parts.

75. Corollary.—Two homologous parts of two straight lines have the same ratio as the two whole lines.

For, AC+CB : EF+FD :: AC : EF (23).
That is, AB : ED :: AC : EF.
Also, AB : ED :: CB : FD.

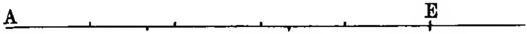
76. Problem in Drawing.—*To find the ratio of two given straight lines.*

Take, for example, the lines b and c.



If these two lines have a common multiple, that is, a line which contains each of them an exact number of times, let x be the number of times that b is contained in the least common multiple of the two lines, and y the number of times it contains c. Then x times b is equal to y times c.

Therefore, from a point A, draw an indefinite straight line AE.



Apply each of the given lines to it a number of times in succession. The ends of the two lines will coincide after x applications of b, and y applications of c.

If the ends coincide for the first time at E, then AE is the least common multiple of the two lines.

The values of x and y may be found by counting, and these express the ratio of the two lines. For since y times c is equal to x times b, it follows that $b : c :: y : x$, which in this case is as 3 to 5.

It may happen that the two lines have no common multiple. In that case the ends will never exactly coincide after any number of applications to the indefinite line; and the ratio can not be exactly expressed by the common numerals.

By this method, however, the ratio may be found within any desired degree of approximation.

77. But this means is liable to all the sources of error that arise from frequent measurements. In practice, it is usual to measure each line as nearly as may be with a comparatively small standard. The numbers thus found express the ratio nearly.

Whenever two lines have any geometrical dependence upon each other, the ratio may be found by calculation with an accuracy which no measurement by the hand can reach.

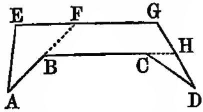
BROKEN LINES.

78. A curve or a broken line is said to be CONCAVE on the side toward the straight line which joins two of its points, and CONVEX to the other side.

79. Theorem.—*A broken line which is convex toward another line that unites its extreme points, is shorter than that line.*

The line ABCD is shorter than the line AEGD, toward which it is convex.

Produce AB and BC till they meet the outer line in F and H.



Since CD is shorter than CHD, it follows (8) that the line ABCD is shorter than ABHD. For a similar reason, ABHD is shorter than AFGD, and AFGD is shorter than AEGD. Therefore, ABCD is shorter than AEGD.

The demonstration would be the same if the outer line were curved, or if it were partly convex to the inner line.

EXERCISE.

80. Vary the above demonstration by producing the lines DC and CB to the left, instead of AB and BC to the right, as in the text; also,

By substituting a curve for the outer line; also,

By letting the inner line consist of two or of four straight lines.

81. A fine thread being tightly stretched, and thus forced to assume that position which is the shortest path between its ends, is a good representation of a straight line. Hence, a stretched cord is used for marking straight lines.

The word *straight* is derived from "*stretch*," of which it is an obsolete participle.

ANGLES.

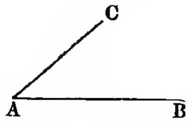
82. An ANGLE is the difference in direction of two lines which have a common point.

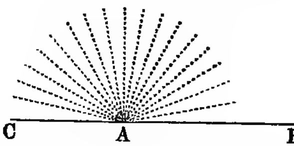
83. Theorem.—*The two lines which form an angle lie in one plane, and determine its position.*

For the plane may pass through the common point and another point in each line, making three in all. These three points determine the position of the plane (60).

DEFINITIONS.

84. Let the line AB be fixed, and the line AC revolve in a plane about the point A; thus taking every direction from A in the plane of its revolution. The angle or difference in direction of the two lines will increase from zero, when AC coincides with AB, till AC takes the direction exactly opposite that of AB.



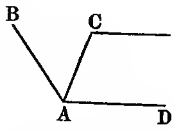


If the motion be continued, AC will, after a complete revolution, again coincide with AB.

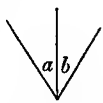
The lines which form an angle are called the SIDES, and the common point is called the VERTEX.

The definition shows that the angle depends upon the directions only, and not upon the length of the sides.

85. Three letters may be used to mark an angle, the one at the vertex being in the middle, as the angle BAC. When there can be no doubt what angle is intended, one letter may answer, as the angle C.



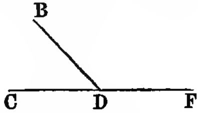
It is frequently convenient to mark angles with letters placed between the sides, as the angles *a* and *b*.



Two angles are ADJACENT when they have the same vertex and one common side between them. Thus, in the last figure, the angles *a* and *b* are adjacent; and, in the previous figure, the angles BAC and CAD.

86. A straight line may be regarded as generated by a point from either end of it, and therefore every straight line has two directions, which are the opposite of each other. We speak of the direction from A to B as the direction AB, and of the direction from B to A as the direction BA.

One line meeting another at some other point than the extremity, makes two angles with it. Thus the angle BDF is the difference in the directions DB and DF; and the angle BDC is the difference in the directions DB and DC.



When two lines pass through or cut each other, four angles are formed, each direction of one line making a difference with each direction of the other.

The opposite angles formed by two lines cutting each other are called VERTICAL angles.

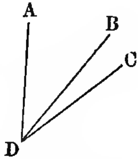
A line which cuts another, or which cuts a figure, is called a SECANT.

PROBLEMS ON ANGLES.

87. Angles may be compared by placing one upon the other, when, if they coincide, they are equal.

Problem.—*One angle may be added to another.*

Let the angles ADB and BDC be adjacent and in the same plane. The angle ADC is plainly equal to the sum of the other two (9).



Problem.—*An angle may be subtracted from a greater one.*

For the angle ADB is the difference between ADC and BDC.

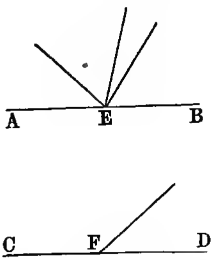
It is equally evident that an angle may be a multiple or a part of another angle; in a word, that angles are quantities which may be compared, added, subtracted, multiplied, or divided.

But angles are not magnitudes, for they have no extent, either linear, superficial, or solid.

ANGLES FORMED AT ONE POINT.

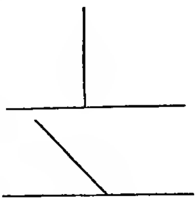
88. Theorem.—*The sum of all the successive angles formed in a plane upon one side of a straight line, is an invariable quantity; that is, all such sums are equal to each other.*

If AB and CD be two straight lines, then the sum of all the successive angles at E is equal to the sum of all those at F.



For the line AE may be placed on CF, the point E on the point F. Then EB will fall on FD, for when two straight lines coincide in part, they must coincide throughout their mutual extent (51). Therefore, the sum of all the angles upon AB exactly coincides with the sum of all the angles upon CD, and the two sums are equal.

89. When one line meets another, making the adjacent angles equal, the angles are called RIGHT ANGLES.



One line is PERPENDICULAR to the other when the angle which they make is a right angle.

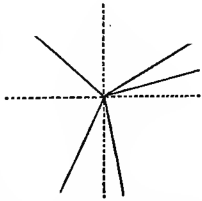
Two lines are OBLIQUE to each other when they make an angle which is greater or less than a right angle.

90. Corollary.—All right angles are equal.

For each is half of the sum of the angles upon one side of a straight line. By the above theorem, these sums are always equal, and (7) the halves of equal quantities are equal.

91. Corollary.—The sum of all the successive angles formed in a plane and upon one side of a straight line, is equal to two right angles.

92. Corollary.—The sum of all the successive angles formed in a plane about a point, is equal to four right angles.



93. Corollary.—When two lines cut each other, if one of the angles thus formed is a right angle, the other three must be right angles.

94. In estimating or measuring angles in geometry, the right angle is taken as the standard.

An angle less than a right angle is called ACUTE.

An angle greater than one right angle and less than the sum of two, is called OBTUSE. Angles greater than the sum of two right angles are rarely used in elementary geometry.

When the sum of two angles is equal to a right angle, each is the COMPLEMENT of the other.

When the sum of two angles is equal to two right angles, each is the SUPPLEMENT of the other.

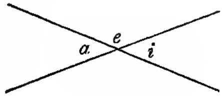
95. Corollary.—Angles which are the complement of the same or of equal angles are equal (7).

96. Corollary.—Angles which are the supplements of the same or of equal angles are equal.

97. Corollary.—The supplement of an obtuse angle is acute.

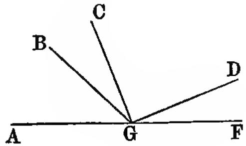
98. Corollary.—The greater an angle, the less is its supplement.

99. Corollary.—Vertical angles are equal. Thus, *a* and *i* are each supplements of *e*.



100. Theorem.—*When the sum of several angles in a plane having their vertices at one point is equal to two right angles, the extreme sides form one straight line.*

If the sum of AGB, BGC, etc., be equal to two right angles, then will AGF be one straight line.



For the sum of all these angles being equal (91) to the sum of the angles upon one side of a straight line, it follows that the two sums may coincide (40), or that AGF may coincide with a straight line. Therefore, AGF is a straight line.

EXERCISES.

101. Which is the greater angle, *a* or *b*, and why?

What is the greatest number of points in which two straight lines may cut each other? In which three may cut each other? Four?

102. The student should ask and answer the question "why" at each step of every demonstration; also, for every corollary. Thus:

Why are vertical angles equal? Why are supplements of the same angles equal?

And in the last theorem: Why is AGF a straight line? Why may AGF coincide with a straight line? Why may the two sums named coincide? Why are the two sums of angles equal?

PERPENDICULAR AND OBLIQUE LINES.

103. Theorem.—*There can be only one line through a given point perpendicular to a given straight line.*

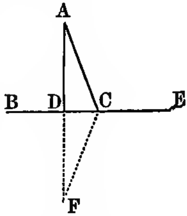
For, since all right angles are equal (90), all lines lying in one plane and perpendicular to a given line, must have the same direction. Now, through a given point in one direction there can be only one straight line (49).

Therefore, since the perpendiculars have the same direction, there can be through a given point only one perpendicular to a given straight line.

When the point is *in* the given line, this theorem must be limited to one plane.

104. Theorem.—*If a perpendicular and oblique lines fall from the same point upon a given straight line, the perpendicular is shorter than any oblique line.*

If AD is perpendicular and AC oblique to BE, then AD is shorter than AC.

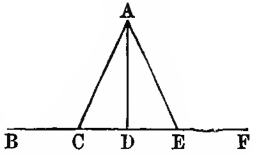


Let the figure revolve upon BE as upon an axis (61), the point A falling upon F, and the lines AD and AC upon FD and FC.

Now, the angle CDF is equal to the angle CDA, and both are right angles. Therefore, the sum of those two angles being equal to two right angles (100), ADF is a straight line, and is shorter than ACF (54). Therefore, AD, the half of ADF, is shorter than AC, the half of ACF.

105. Corollary.—The distance from a point to a straight line is the perpendicular let fall from the point to the line.

106. Theorem.—*If a perpendicular and several oblique lines fall from the same point upon a given straight line, and if two oblique lines meet the given line at equal distances from the foot of the perpendicular, the two are equal.*



Let AD be the perpendicular and AC and AE the oblique lines, making CD equal to DE. Then AC and AE are equal.

Let that portion of the figure on the left of AD turn upon AD. Since the angles ADB and ADF are equal, DB will take the direction DF; and since DC and DE are equal, the point C will fall on E. Therefore, AC and AE will coincide (51), and are equal.

107. Corollary.—When the oblique lines are equal, the angles which they make with the perpendicular are equal. For CAD may coincide with DAE.

108. Theorem.—*If a line be perpendicular to another at its center, then every point of the perpendicular is equally distant from the two ends of the other line.*

For straight lines extending from any point of the perpendicular to the two ends of the other line must be equal (106).

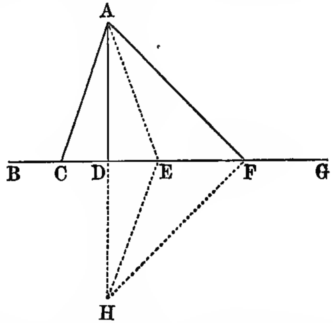
Let the student make a diagram of this. Then state what lines are given by the hypothesis, and what are constructed for demonstration.

109. Corollary.—Since two points fix the position of a line, if a line have two points each equidistant from the ends of another line, the two lines are perpendicular to each other, and the second line is bisected.

The two points may be on the same side, or on opposite sides of the second line.

110. Theorem.—*If a perpendicular and several oblique lines fall from the same point on a given straight line, of two oblique lines, that which meets the given line at a greater distance from the perpendicular is the longer.*

If AD be perpendicular to BG, and DF is greater than DC, then AF is greater than AC.



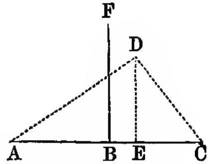
On the line DF take a part DE equal to DC, and join AE. Then let the figure revolve upon BG, the point A falling upon H, and the lines AD, AE, and AF upon HD, HE, and HF.

Now, AEH is shorter than AFH (79); therefore, AE, the half of AEH, is shorter than AF, the half of AFH. But AC is equal to AE (106). Hence, AF is longer than AC, or AE, or any line from A meeting the given line at a less distance from D than DF.

111. Corollary.—A point may be at the same distance from two points of a straight line, one on each side of the perpendicular; but it can not be at the same distance from more than two points.

112. Theorem.—*If a line be perpendicular to another at its center, every point out of the perpendicular is nearer to that end of the line which is on the same side of the perpendicular.*

If BF is perpendicular to AC at its center B, then D, a point not in BF, is nearer to C than to A.



Join DA and DC, and let the perpendicular DE fall from D upon the line AC.

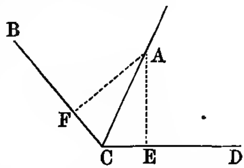
This perpendicular must fall on the same side of BF as the point D, for if it crossed the line BF, there would be from the point of intersection two perpendiculars on AC, which is impossible (103). Now, since AB is equal to BC, AE must be greater than EC. Hence, AD is greater than CD (110).

The point D is supposed to be in the plane of ACF. If it were not, the perpendicular from it might fall on the point B.

BISECTED ANGLE.

113. Theorem.—*Every point of the line which bisects an angle is equidistant from the sides of the angle.*

Let BCD be the given angle, and AC the bisecting line. Then the distance of the two sides from any point A of that line is measured by perpendiculars to the sides, as AF and AE.



Since the angles BCA and DCA are equal, that part of the figure upon the one side of AC may revolve upon AC, and the line BC will take the direction of CD, and coincide with it.

Then the perpendiculars AF and AE must coincide (103), and the point F fall upon E. Therefore, AF and AE are equal, and the point A is equally distant (105) from the sides of the given angle.

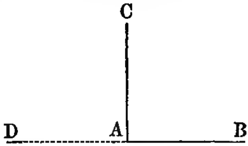
APPLICATION.

114. Perpendicular lines are constantly used in architecture, carpentry, stone-cutting, machinery, etc.

The mason's square consists of two flat rulers made of iron, and connected together in such a manner that both edges of one are at right angles to those of the other. The carpenter's square is much like it, but one of the legs is wood. This instrument is used for drawing perpendicular lines, and for testing the correctness of right angles.

The square itself should be tested in the following manner:

On any plane surface draw an angle, as BAC, with the square. Extend BA in the same straight line to D. Then turn the square so that the edges by which the angle BAC was described, may be applied to the angle DAC. If the coincidence is exact, the square is correct as to these edges.



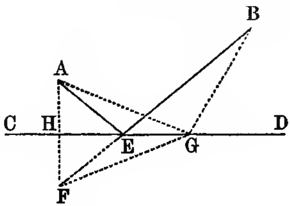
Let the student show that this method of testing the square is according to geometrical principles.

The square here described is not the geometrical figure of that name, which will be defined hereafter.

A MINIMUM LINE.

115. Theorem.—*Of any two lines which may extend from two given points outside of a straight line to any point in it, those which are together least make equal angles with that line.*

Let CD be the line and A and B the points, and AEB the shortest line that can be made from A to B through any point of CD. Then it is to be proved that AEC and BED are equal angles.



Make AH perpendicular to CD, and produce it to F, making HF equal to AH.

Now every point of the line CD is equally distant from A and F (108). Therefore, every line joining B to

F through some point of CD, is equal to a line joining B to A through the same point. Thus, BGF is equal to BGA, since GF and GA are equal. So, BEF is equal to BEA.

But BEA is, by hypothesis, the shortest line from B to A through any point of CD. Therefore, BEF is the shortest line from B to F, and is a straight line (54).

Since BEF is one straight line, the angles FEH and BED are vertical and equal (99). But the angles FEH and AEH are equal (107). Therefore, AEH and BED are equal (6).

116. When several magnitudes are of the same kind but vary in extent, the least is called a *minimum*, and the greatest a *maximum*.

APPLICATION.

When a ray of light is reflected from a polished surface, the incident and reflected parts of the ray make equal angles with the surface. We learn from this geometrical principle that light, when reflected, still adheres to that law of its nature which requires it to take the shortest path.

PARALLELS.

117. Parallel lines are straight lines which have the same directions.

118. Corollary.—Two lines which are each parallel to a third are parallel to each other.

119. Corollary.—From the above definition, and the *Axiom of Direction* (49), it follows that there can be only one line through a given point parallel to a given line.

120. Corollary.—From the same premises, it follows that two parallel lines can never meet, or have a common point.

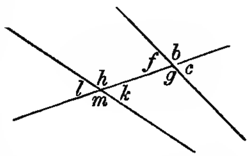
121. Theorem.—*Two parallel lines both lie in one plane and determine its position.*

The position of a plane is determined (60) by either line and one point of the other line. Now the plane has the direction of the first line and can not vary from it (56), and the second line has also the same direction (117) and can not vary from it (44).

Therefore, the second line must also lie wholly in the plane.

NAMES OF ANGLES.

122. When two straight lines are cut by a secant, the eight angles thus formed are named as follows:



The four angles between the two lines are INTERIOR; as, *f*, *g*, *h*, and *k*. The other four are EXTERIOR; as, *b*, *c*, *l*, and *m*.

Two angles on the same side of the secant, and on the same side of the two lines cut by it, are called CORRESPONDING angles. The angles *h* and *b* are corresponding.

Two angles on opposite sides of the secant, and on opposite sides of the two lines cut by it, are called ALTERNATE angles. The angles *f* and *k* are alternate; also, *b* and *m*.

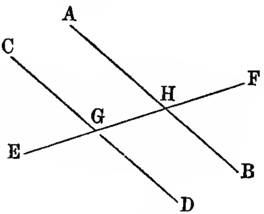
The student should name the corresponding and the alternate angles of each of the eight angles in the above diagram. Let him also name them in the diagram of the following theorem.

123. Corollary.—The corresponding and the alternate angles of any given angle are vertical to each other, and therefore equal (99).

PARALLELS CUT BY A SECANT.

124. Theorem.—*When two parallel lines are cut by a secant, each of the eight angles is equal to its corresponding angle.*

If the straight lines AB and CD have the same directions, then the angles FHB and FGD are equal.



For, since the directions GD and HB are the same, the direction GF differs equally from them. Therefore, the angles are equal (82).

In the same manner, it may be shown that any two corresponding angles are equal.

125. Corollary.—When two parallel lines are cut by a secant, each of the eight angles is equal to its alternate (123).

126. Corollary.—Two interior angles on the same side of the secant are supplements of each other. For, since GHB is the supplement of FHB (91), it is also the supplement of its equal HGD. Two exterior angles on the same side of the secant are supplementary, for a similar reason.

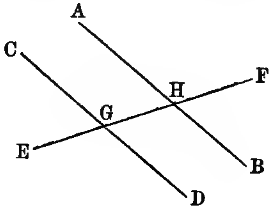
127. Corollary.—When a secant is perpendicular to one of two parallels, it is also perpendicular to the other, and all the angles are right.

Let the student illustrate by a diagram, in this and in all cases when a diagram is not given.

128. Corollary.—When the secant is oblique to the parallels, four of the angles formed are obtuse and are equal to each other; the other four are acute, and equal; and any acute angle is the supplement of any obtuse.

129. Theorem.—*When two straight lines, being in the same plane, are cut by a third, making the corresponding angles equal, the two lines so cut are parallel.*

If AB and CD lie in the same plane, and if the angles AHF and CGF are equal, then AB and CD are parallel.



For, suppose a straight line to pass through the point H, parallel to DC. Such a line makes a corresponding angle equal to CGF, and therefore equal to AHF. This supposed parallel line lies in the same plane as CD and H (121); that is, by hypothesis, in the same plane as AB. But if it lies in the same plane with AB and makes the same angle with the same line EF, at the same point H, then it must coincide with AB. For, when two angles are equal and placed one upon the other, they coincide throughout. Therefore, AB is parallel to CD.

130. Corollary.—If the alternate angles are equal, the lines are parallel (123).

131. Corollary.—The same conclusion must follow when the interior angles on the same side of the secant are supplementary.

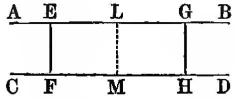
DISTANCE BETWEEN PARALLELS.

132. Theorem.—*Two parallel lines are everywhere equally distant.*

The distance between two parallel lines is measured by a line perpendicular to them, since it is the shortest from one to the other.

Let AB and CD be two parallels. Then any perpendiculars to them, as EF and GH, are equal. From M, the center of FH, erect the perpendicular ML.

Let that part of the figure to the left of ML revolve upon ML. All the angles of the figure being right angles, MC will fall upon MD. Since MF is equal to MH, the point F will fall on H, and the angles at F and H being equal, FE will take the direction HG, and the point E will be on the line HG. But since the angles at L are equal, the point E will also fall on LB, and being on both LB and HG, it must be on G. Therefore, FE and HG coincide and are equal.



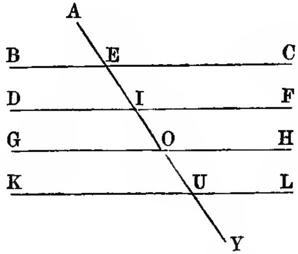
133. Corollary.—The parts of parallel lines included between perpendiculars to them, must be equal. For the perpendiculars are parallel (129).

SECANT AND PARALLELS.

134. Theorem.—*If several equally distant parallel lines be cut by a secant, the secant will be divided into equal parts.*

If the parallels BC, DF, GH, and KL are at equal distances, then the parts EI, IO, and OU of the secant AY are equal.

For that part of the figure included between BC and DF may be placed upon and will coincide with that part between DF and GH; for the parallels are everywhere equally distant (132).



Let them be so placed that the point E may fall upon I. Then, since the angles BEI and DIO (124) are equal, the line EI will take the direction IO. And since DF and GH coincide, the point I will fall on O. Therefore, EI and IO coincide and are equal. In like manner, show that any two of the intercepted parts of the line AY are equal.

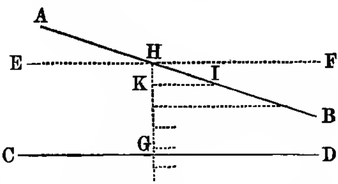
135. Corollary.—Conversely, if several parallel lines intercept equal segments of a secant, then the several distances between the parallels are equal.

136. Corollary.—When the distances between the parallels are unequal, the segments of the secant are unequal. And conversely, when the segments of the secant are unequal, the distances are unequal.

LINES NOT PARALLEL MEET.

137. Theorem.—*If two straight lines are in the same plane and are not parallel, they will meet if sufficiently produced.*

Let AB and CD be two lines. Let the line EF, parallel to CD, pass through any point of AB, as H. From H let the perpendicular HG fall upon CD.



Since AB and EF have different directions, they cut each other at the point H. Take any point, as I, in that part of AB which lies between EF and CD, and extend a line IK parallel to CD through the point I. Now divide HG into parts equal to HK until one of the points of division falls beyond G. Then along HB, take parts equal to HI, as often as

HK was taken along HG. Lastly, from each point of division of HB, extend a line perpendicular to HG.

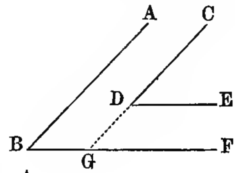
These perpendiculars are parallel to each other and to CD (129). These parallels by construction intercept equal parts of HB. Therefore (135), they are equally distant from each other. Hence, HG is divided by them into equal segments (134); that is, each one passes through one of the previously ascertained points of the line HG.

But the last of these points was beyond the line CD, and as the parallel can not cross CD (120), the corresponding point of HB is beyond CD. Therefore, HB and CD must cross each other.

ANGLES WITH PARALLEL SIDES.

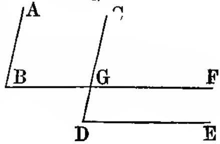
138. Theorem.—*When the sides of one angle are parallel to the sides of another, and have respectively the same directions from their vertices, the two angles are equal.*

If the directions BA and DC are the same, and the directions DE and BF are the same, then the angles ABF and CDE are equal.



For each of these angles is equal to the angle CGF (124).

139. Let the student demonstrate that when two of the parallel sides have opposite directions, and the other two have the same direction, then the angles are supplementary.

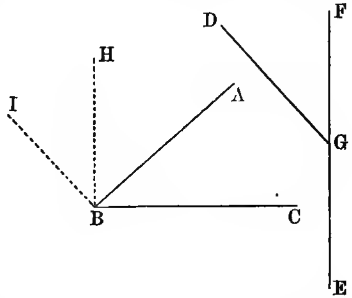


Let him also demonstrate that if both sides of one angle have directions respectively opposite to those of the other, then again the angles are equal.

ANGLES WITH PERPENDICULAR SIDES.

140. Theorem.—*Two angles which have their sides respectively perpendicular are equal or supplementary.*

If AB is perpendicular to DG, and BC is perpendicular to EF, then the angle ABC is equal to one, and supplementary to the other of the angles formed by DG and EF (86).

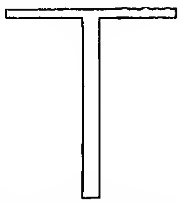


Through B extend BI parallel to GD, and BH parallel to EF.

Now, ABI and CBH are right angles (127), and therefore equal (90). Subtracting the angle HBA from each, the remainders HBI and ABC are equal (7). But HBI is equal to FGD (138), and is the supplement of EGD (139). Therefore, the angle ABC is equal or supplementary to any angle formed by the lines DG and EF.

APPLICATIONS.

141. The instrument called the T square consists of two straight and flat rulers fixed at right angles to each other, as in the figure. It is used to draw parallel lines.



Draw a straight line in a direction perpendicular to that in which it is required to draw parallel lines. Lay the cross-piece of the T ruler along this line. The other piece of the ruler gives the direction of one of the parallels. The ruler being moved along the paper, keeping the cross-piece coincident with the line first described, any number of parallel lines may be drawn.

What is the principle of geometry involved in the use of this instrument?

142. The uniform distance of parallel lines is the principle upon which numerous instruments and processes in the arts are founded.

If two systems, each consisting of several parallel lines, cross each other at right angles, all the parts of one system included between any two lines of the other system will be equal. The ordinary framing of a window consists of two systems of lines of this kind; the shelves and upright standards of book-cases and the paneling of doors also afford similar examples.

143. The joiner's gauge is a tool with which a line may be drawn on a board parallel to its edge. It consists of a square piece of wood, with a sharp steel point near the end of one side, and a movable band, which may be fastened by a screw or key at any required distance from the point. The gauge is held perpendicular to the edge of the board, against which the band is pressed while the tool is moved along the board, the steel point tracing the parallel line.

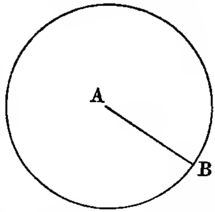
144. It is frequently important in machinery that a body shall have what is called a parallel motion; that is, such that all its parts shall move in parallel lines, preserving the same relative position to each other.

The piston of a steam-engine, and the rod which it drives, receive such a motion; and any deviation from it would be attended with consequences injurious to the machinery. The whole mass of the piston and its rod must be so moved, that every point of it shall describe a line exactly parallel to the direction of the cylinder.

CHAPTER IV.

THE CIRCUMFERENCE.

145. Let the line AB revolve in a plane about the end A, which is fixed. Then the point B will describe a line which returns upon itself, called a circumference of a circle. Hence, the following definitions:



A CIRCLE is a portion of a plane bounded by a line called a CIRCUMFERENCE, every point of which is equally distant from a point within called the CENTER.

146. Theorem.—*A circumference is curved throughout.*

For a straight line can not have more than two points equally distant from a given point (111).

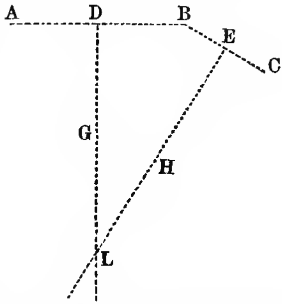
147. Corollary.—A straight line can not cut a circumference in more than two points.

148. The circumference is the only curve considered in elementary geometry. Let us examine the properties of this line, and of the straight lines which may be combined with it.

HOW DETERMINED.

149. Theorem.—*Three points not in the same straight line fix a circumference both as to position and extent.*

The three given points, as A, B, and C, determine the position of a plane. Let the given points be joined by straight lines AB and BC. At D and E, the middle points of these lines, let perpendiculars be erected in the plane of the three points.



By the hypothesis, AB and BC make an angle at B. Therefore, GD is not perpendicular to BC, for if it were, AB and BC would be parallel (129). Hence, DG and EH are not parallel (117), since one is perpendicular and the other is not perpendicular to BC. Therefore, DG and EH will meet (137) if produced. Let L be their point of intersection.

Since every point of DG is equidistant from A and B (108), and since every point of EH is equidistant from B and C, their common point L is equidistant from A, B, and C. Therefore, with this point as a center, a circumference may be described through A, B, and C. There can be no other circumference through these three points, for there is no other point besides L equally distant from all three (112).

Therefore, these three points fix the position and the extent of the circumference which passes through them.

ARCS AND RADII.

150. An Arc is a portion of a circumference.

A Radius is a straight line from the center to the circumference.

A Diameter is a straight line passing through the center, and limited at both ends by the circumference.

A Chord is a straight line joining the ends of an arc.

151. Corollary.—All radii of the same circumference are equal.

152. Corollary.—In the same circumference, a diameter is double the radius, and all diameters are equal.

153. Corollary.—Every point of the plane at greater distance from the center than the length of the radius, is outside of the circumference. Every point at a less distance from the center, is within the circumference. Every point whose distance from the center is equal to the radius, is on the circumference.

154. Theorem.—*Circumferences which have equal radii are equal.*

Let the center of one be placed on that of the other. Then the circumferences will coincide. For if it were otherwise, then some points would be unequally distant from the common center, which is impossible when the radii are equal. Therefore, the circumferences are equal.

155. Corollary.—A circumference may revolve upon, or slide along its equal.

156. Corollary.—Two arcs of the same or of equal circles may coincide so far as both extend.

157. Theorem.—*Every diameter bisects the circumference and the circle.*

For that part upon one side of the diameter may be turned upon that line as its axis. When the two parts thus meet, they will coincide; for if they did not, some points of the circumference would be unequally distant from the center.

158. A line which divides any figure in this manner, is said to divide it *symmetrically;* and a figure which can be so divided is *symmetrical.*

159. Theorem.—*A diameter is greater than any other chord of the same circumference.*

To be demonstrated by the student.

160. Problem.—*Arcs of equal radii may be added together, or one may be subtracted from another.*

For an arc may be produced till it becomes an entire circumference, or it may be diminished at will (35 and 145).

Therefore, the length of an arc may be increased or decreased by the length of another arc of the same radius; and the result, that is, the sum or difference, will be an arc of the same radius.

161. Corollary.—Arcs of equal radii may be multiplied or divided in the same manner as straight lines.

162. The sum of several arcs may be greater than a circumference.

163. Two arcs not having the same radius may be joined together, and the result may be called their sum; but it is not one arc, for it is not a part of one circumference.

APPLICATIONS.

164. The circumference is the only line which can move along itself, around a center, without suffering any change. For any line that can do this must, therefore, have all its points equally distant from the center of revolution; that is, it must be a circumference.

It is in virtue of this property that the axles of wheels, shafts, and other solid bodies which are required to revolve within a hollow mold or casing of their own form, must be circular. If they were of any other form, they would be incapable of revolving without carrying the mold or casing around with them.

165. Wheels which are intended to maintain a carriage always at the same hight above the road on which they roll, must be circular, with the axle in the center.

166. The art of turning consists in the production of the circular form by mechanical means. The substance to be turned is placed in a machine called a lathe, which gives it a rotary motion. The edge of a cutting tool is placed at a distance from the axis of revolution equal to the radius of the intended circle. As the substance revolves, the tool removes every part that is further from the axis than the radius, and thus gives a circular form to what remains.

PROBLEMS IN DRAWING.

167. The compasses enable us to draw a circumference, or an arc of a given radius and given center.

Open the instrument till the points are on the two ends of the given radius. Then fix one point on the given center, and the other point may be made to revolve around in contact with the surface, thus tracing out the circumference.

The revolving leg may have a pen or pencil at the point. In the operation, care should be taken not to vary the opening of the compasses.

168. It is evident that with the ruler and compasses (69),

1. A straight line can be drawn through two given points.
2. A given straight line can be produced any length.
3. A circumference can be described from any center, with any radius.

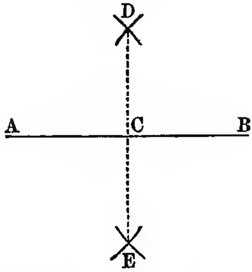
169. The foregoing are the three postulates of Euclid. Since the straight line and the circumference are the only lines treated of in elementary geometry, these Euclidian postulates are a sufficient basis for all problems. Hence, the rule that no instruments shall be used except the ruler and the compasses (68).

170. In the Elements of Euclid, which, for many ages, was the only text-book on elementary geometry, the problems in drawing occupy the place of problems in geometry. At present, the mathematicians of Germany, France, and America put them aside as not forming a necessary part of the theory of the science. English writers, however, generally adhere to Euclid.

171. Problem.—*To bisect a given straight line.*

With A and B as centers, and with a radius greater than the half of AB, describe arcs which intersect in the two points D and E. The straight line joining these two points will bisect AB at C.

Let the demonstration be given by the student (109 and 151).

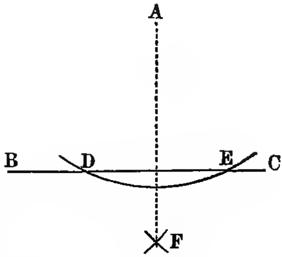


172. Problem.—*To erect a perpendicular on a given straight line at a given point.*

Take two points in the line, one on each side of the given point, at equal distances from it. Describe arcs as in the last problem, and their intersection gives one point of the perpendicular.

Demonstration to be given by the student.

173. Problem.—*To let fall a perpendicular from a given point on a given straight line.*



With the given point as a center, and a radius long enough, describe an arc cutting the given line BC in the points D and E. The line may be produced, if necessary, to be cut by the arc in two places. With D and E as centers, and with a radius greater than the half of DE, describe arcs cutting each other in F. The straight line joining A and F is perpendicular to DE.

Let the student show why.

174. Problem.—*To draw a line through a given point parallel to a given line.*

Let a perpendicular fall from the point on the line. Then, at the given point, erect a perpendicular to this last. It will be parallel to the given line.

Let the student explain why (129).

175. Problem.—*To describe a circumference through three given points.*

The solution of this problem is evident, from Article 149.

176. Problem.—*To find the center of a given arc or circumference.*

Take any three points of the arc, and proceed as in the last problem.

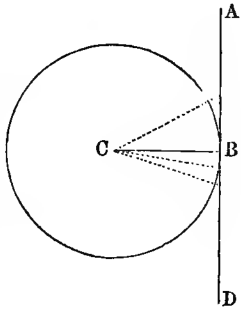
177. The student is advised to make a drawing of every problem. First draw the parts given, then the construction requisite for solution. Afterward demonstrate its correctness.

Endeavor to make the drawing as exact as possible. Let the lines be fine and even, as they better represent the abstract lines of geometry. A geometrical principle is more easily understood by the student, when he makes a neat diagram, than when his drawing is careless.

TANGENT.

178. Theorem.—*A straight line which is perpendicular to a radius at its extremity, touches the circumference in only one point.*

Let AD be perpendicular to the radius BC at its extremity B. Then it is to be proved that AD touches the circumference at B, and at no other point.



If the center C be joined by straight lines with any points of AD, the perpendicular BC will be shorter than any such oblique line (104). Therefore (153), every point of the line AD, except B, is outside of the circumference.

179. A TANGENT is a line touching a circumference in only one point. The circumference is also said to be tangent to the straight line. The common point is called the *point of contact.*

APPLICATION.

180. Tangent lines are frequently used in the arts. A common example is when a strap is carried round a part of the circumference of a wheel, and extending to a distance, sufficient tension is given to it to produce such a degree of friction between it and the wheel, that one can not move without the other.

181. Problem in Drawing.—*To draw a tangent at a given point of an arc.*

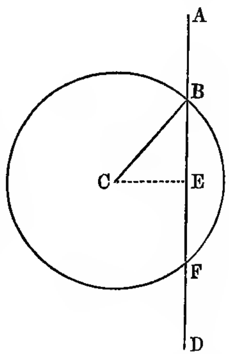
Draw a radius to the given point, and erect a perpendicular to the radius at that point.

It will be necessary to produce the radius beyond the arc, as the student has not yet learned to erect a perpendicular at the extremity of a line without producing it.

SECANT.

182. Theorem.—*A straight line which is oblique to a radius at its extremity, cuts the circumference in two points.*

Let AD be oblique to the radius CB at its extremity B. Then it will cut the circumference at B, and at some other point.



From the center C, let CE fall perpendicularly on AD. On ED, take EF equal to EB.

Then the distance from C to any point of the line AD between B and F is less than the length of the radius CB (110), and to any point of the line beyond B and F, it is greater than the length of CB. Therefore (153), that portion of the line AD between B and F is within, and the parts beyond B and F are without the circumference. Hence, the oblique line cuts the circumference in two points.

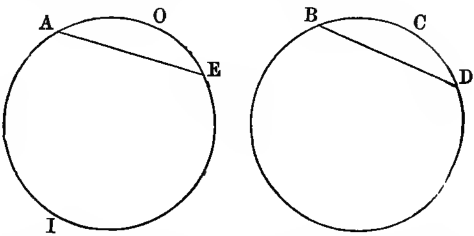
183. Corollary.—A tangent to the circumference is perpendicular to the radius which extends to the point of contact. For, if it were not perpendicular, it would be a secant.

184. Corollary.—At one point of a circumference, there can be only one tangent (103).

CHORDS.

185. Theorem.—*The radii being equal, if two arcs are equal their chords are also equal.*

If the arcs AOE and BCD are equal, and their radii are equal, then AE and BD are equal.



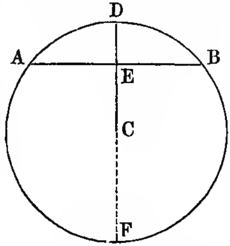
For, since the radii are equal, the circumferences are equal (154); and the arcs may be placed one upon the other, and will coincide, so that A will be upon B, and E upon D. Then the two chords, being straight lines, must coincide (51), and are equal.

186. Every chord subtends two arcs, which together form the whole circumference. Thus the chord AE subtends the arcs AOE and AIE.

The arc of a chord always means the smaller of the two, unless otherwise expressed.

187. Theorem.—*The radius which is perpendicular to a chord bisects the chord and its arc.*

Let CD be perpendicular at E to the chord AB, then will AE be equal to EB, and the arc AD to the arc DB.



Produce DC to the circumference at F, and let that part of the figure on one side of DF be turned upon DF as upon an axis. Then the semi-circumference DAF will coincide with DBF (157). Since the angles at E are right, the line EA will take the direction of EB, and the point A will fall on the point B. Therefore, EA and EB will coincide, and are equal; and the same is true of DA and DB, and of FA and FB.

188. Corollary.—Since two conditions determine the position of a straight line (52), if it has any two of the four conditions mentioned in the theorem, it must have the other two. These four conditions are,

1. The line passes through the center of the circle, that is, it is a radius.
2. It passes through the center of the chord.
3. It passes through the center of the arc.
4. It is perpendicular to the chord.

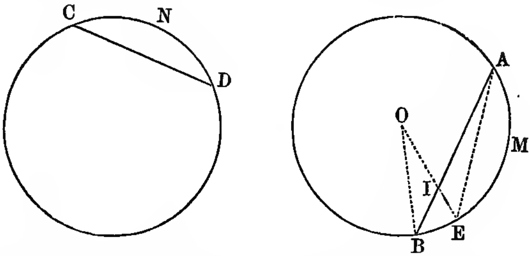
189. Theorem.—*The radii being equal, when two arcs are each less than a semi-circumference, the greater arc has the greater chord.*

If the arc AMB is greater than CND, and the radii of the circles are equal, then AB is greater than CD.

Take AME equal to CND. Join AE, OE, and OB. Then AE is equal to CD (185).

Since the arc AMB is less than a semi-circumference, the chord AB will pass between the arc and the center O, Hence, it cuts the radius OE at some point I.

Now, the broken line OIB is greater than OB (54), or its equal OE. Subtracting OI from each (8), the



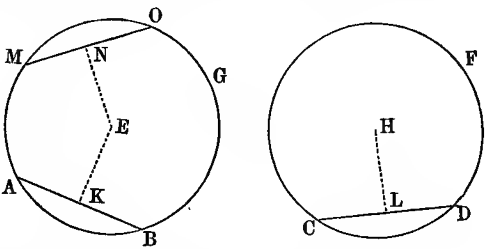
remainder IB is greater than the remainder IE. Adding AI to each of these, we have AB greater than AIE. But AIE is greater than AE. Therefore, AB, the chord of the greater arc, is greater than AE, or its equal CD, the chord of the less.

190. Corollary.—When the arcs are both greater than a semi-circumference, the greater arc has the less chord.

DISTANCE FROM THE CENTER.

191. Theorem.—*When the radii are equal, equal chords are equally distant from the center.*

Let the chords AB and CD be equal, and in the equal



circles ABG and CDF; then the distances of these chords from the centers E and H will also be equal.

Let fall the perpendiculars EK and HL from the centers upon the chords.

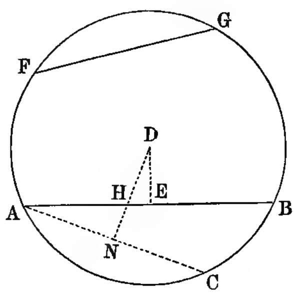
Now, since the chords AB and CD are equal, the arcs AB and CD are also equal (185); and we may apply the circle ABG to its equal CDF, so that they will coincide, and the arc AB coincide with its equal CD. Therefore, the chords will coincide. Since K and L are the middle points of these coinciding chords (187), K will fall upon L. Therefore, the lines EK and HL coincide and are equal. But these equal perpendiculars measure the distance of the chords from the centers (105).

If the equal chords, as MO and AB, are in the same circle, each may be compared with the equal chord CD of the equal circle CDF.

Thus it may be proved that the distances NE and EK are each equal to HL, and therefore equal to each other.

192. Theorem.—*When the radii are equal, the less of two unequal chords is the farther from the center.*

Let AB be the greater of two chords, and FG the less, in the same or an equal circle. Then FG is farther from the center than AB.



Take the arc AC equal to FG. Join AC, and from the center D let fall the perpendiculars DE and DN upon AB and AC.

Since the arc AC is less than AB, the chord AB will be between AC and the center D, and will cut the perpendicular DN. Then DN, the whole, is greater than DH, the part cut off; and DH is greater than DE (104). So much the more is DN greater than DE. Therefore, AC and its equal FG are farther from the center than AB.

193. Corollary.—Conversely of these two theorems, when the radii are equal, chords which are equally distant from the center are equal; and of two chords which are unequally distant from the center, the one nearer to the center is longer than the other.

194. Problem in Drawing.—*To bisect a given arc.*

Draw the chord of the arc, and erect a perpendicular at its center.

State the theorem and the problems in drawing here used.

195. "The most simple case of the division of an arc, after its bisection, is its trisection, or its division into three equal parts. This problem accordingly exercised, at an early epoch in the progress of geometrical science, the ingenuity of mathematicians, and has become memorable in the history of geometrical discovery, for having baffled the skill of the most illustrious geometers.

"Its object was to determine means of dividing any given arc into three equal parts, without any other instruments than the rule and compasses permitted by the postulates prefixed to Euclid's Elements. Simple as the problem appears to be, it never has been solved, and probably never will be, under the above conditions." —*Lardner's Treatise.*

ANGLES AT THE CENTER.

196. Angles which have their vertex at the center of a circle are called, for this reason, *angles at the center.* The arc between the sides of an angle is called the *intercepted arc of the angle.*

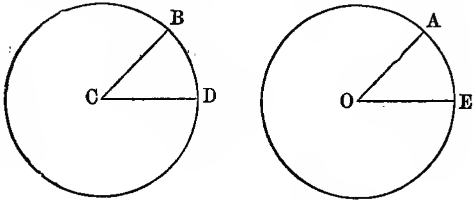
197. Theorem.—*The radii being equal, any two angles at the center have the same ratio as their intercepted arcs.*

This theorem presents the three following cases:

1st. If the arcs are equal, the angles are equal.

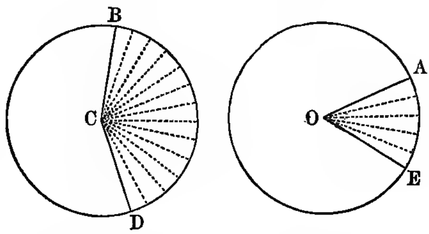
For the arcs may be placed one upon the other, and will coincide. Then BC will coincide with AO, and DC with EO. Thus the angles may coincide, and are equal.

The converse is proved in the same manner.



2d. If the arcs have the ratio of two whole numbers, the angles have the same ratio.

Suppose, for example, the arc BD : arc AE : : 13 : 5.



Then, if the arc BD be divided into thirteen equal parts, and the arc AE into five equal parts, these small arcs will all be equal. Let radii join to their respective centers all the points of division.

The small angles at the center thus formed are all equal, because their intercepted arcs are equal. But BCD is the sum of thirteen, and AOE of five of these equal angles. Therefore,

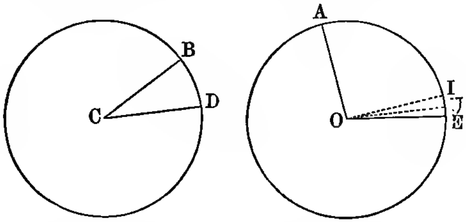
angle BCD : angle AOE : : 13 : 5;

that is, the angles have the same ratio as the arcs.

3d. It remains to be proved, that, if the ratio of the arcs can not be expressed by two whole numbers, the angles have still the same ratio as the arcs; or, that the radius being the same, the

arc BD : arc AE : : angle BCD : angle AOE.

If this proportion is not true, then the first, second,



and third terms being unchanged, the fourth term is either too large or too small. We will prove that it is neither. If it were too large, then some smaller angle, as AOI, would verify the proportion, and

arc BD : arc AE : : angle BCD : angle AOI.

Let the arc BD be divided into equal parts, so small that each of them shall be less than EI. Let one of these parts be applied to the arc AE, beginning at A, and marking the points of division. One of those points must necessarily fall between I and E, say at the point U. Join OU.

Now, by this construction, the arcs BD and AU have the ratio of two whole numbers. Therefore,

arc BD : arc AU : : angle BCD : angle AOU.

These last two proportions may be written thus (19):

arc BD : angle BCD : : arc AE : angle AOI;

arc BD : angle BCD : : arc AU : angle AOU.

Therefore (21),

arc AE : angle AOI : : arc AU : angle AOU;

or (19),

arc AE : arc AU : : angle AOI : angle AOU.

But this last proportion is impossible, for the first antecedent is greater than its consequent, while the second antecedent is less than its consequent. Therefore, the supposition which led to this conclusion is false, and the fourth term of the proportion, first stated, is not too large. It may be shown, in the same way, that it is not too small.

Therefore, the angle AOE is the true fourth term of the proportion, and it is proved that the arc BD is to the arc AE as the angle BCD is to the angle AOE.

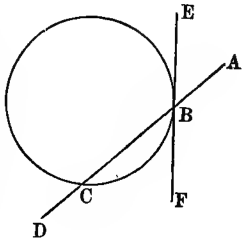
DEMONSTRATION BY LIMITS.

198. The third case of the above proposition may be demonstrated in a different manner, which requires some explanation.

We have this definition of a limit: Let a magnitude vary according to a certain law which causes it to approximate some determinate magnitude. Suppose the first magnitude can, by this law, approach the second indefinitely, but can never quite reach it. Then the second, or invariable magnitude, is said to be the *limit* of the first, or variable one.

199. Any curve may be treated as a limit. The straight parts of a broken line, having all its vertices in the curve, may be diminished at will, and the broken line made to approximate the curve indefinitely. Hence, a curve is the limit of those broken lines which have all their vertices in the curve.

200. The arc BC, which is cut off by the secant AD, may be diminished by successive bisections, keeping the remainders next to B. Thus AD, revolving on the point B, may approach indefinitely the tangent EF. Hence, the tangent at any point of a curve is the limit of the secants which may cut the curve at that point.

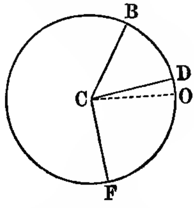


201. The principle upon which all reasoning by the method of limits is governed, is that, *whatever is true up to the limit is true at the limit.* We admit this as an axiom of reasoning, because we can not conceive it to be otherwise.

Whatever is true of every broken line having its vertices in a curve, is true of that curve also. Whatever is true of every secant passing through a point of a curve, is true of the tangent at that point.

We do not say that the arc is a broken line, nor that the tangent is a secant, nor that an arc can be without extent; but that the curve and the tangent are limits toward which variable magnitudes may tend, and that whatever is true all the way to within the least possible distance of a certain point, is true at that point.

202. Having proved (first and second parts, 197) that, when two arcs have the ratio of two whole numbers, the angles at the center have the same ratio, we may then suppose that the ratio of BD to BF can not be expressed by whole numbers.



Now, if we divide BF into two equal parts, the point of division will be at a certain distance from D. We may conceive the arc BF to be divided into any number of equal parts, and by increasing this number, the point O, the point of division nearest to D, may be made to approach within any conceivable distance of D. By the second part of the theorem (197), it is proved that

arc BO : arc BF :: angle BCO : angle BCF.

Now, although the arc BD is itself incommensurable with BF, yet it is the limit of the arcs BO, and the angle BCD is the limit of the angles BCO. Therefore, since whatever is true up to the limit is true at the limit,

arc BD : arc BF :: angle BCD : angle BCF.

That is, the intercepted arcs have the same ratio as their angles at the center.

METHOD OF INFINITES.

203. Modern geometers have made much use of a kind of reasoning which may be called *the method of infinites.* It consists in supposing that any line of definite extent and form is composed of an infinite number of infinitely small straight lines.

A surface is supposed to consist of an infinite number of infinitely narrow surfaces, and a solid of an infinite number of infinitely thin solids. These thin solids, narrow surfaces, and small lines, are called *infinitesimals.*

204. The reasoning of the method of infinites is substantially the same in its logical rigor as of the method of limits. The method of infinites is a much abbreviated form of the method of limits.

The student must be careful how he adopts it. For when the infinite is brought into an argument by the unskillful, the conclusion is very apt to be absurd. It is sufficient to say, that where the method of limits can be used, the method of infinites may also be used without error.

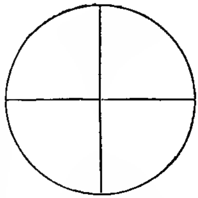
The method of infinites has also been called the *method of indivisibles.* Some examples of its use will be given in the course of the work.

ARCS AND ANGLES.

We return to the subject of angles at the center. The theorem last given (197) has the following

205. Corollary.—If two diameters are perpendicular to each other, they divide the whole circumference into four equal parts.

206. A QUADRANT is the fourth part of a circumference.



207. Since the angle at the center varies as the intercepted arc, mathematicians have adopted the same method of measuring both angles and arcs. As a right angle is the unit of angles, so a quadrant of a certain radius may be taken as the standard for the measurement of arcs that have the same radius.

For the same reason, we usually say that the intercepted arc measures the angle at the center. Thus, the right angle is said to be measured by the quadrant; half a right angle, by one-eighth of a circumference; and so on.

APPLICATIONS.

208. In the applications of geometry to practical purposes, the quadrant and the right angle are divided into ninety equal parts, each of which is called a degree. Each degree is marked thus °, and is divided into sixty minutes, marked thus ′; and each minute is divided into sixty seconds, marked thus ″.

Hence, it appears that there are in an entire circumference, or in the sum of all the successive angles about a point, 360°, or 21 600′, or 1 296 000″. Some astronomers, mostly the French, divide the right angle and the quadrant into one hundred parts, each of these into one hundred; and so on.

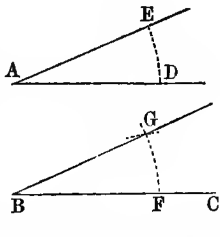
209. Instruments for measuring angles are founded upon the principle that arcs are proportional to angles. Such instruments usually consist either of a part or an entire circle of metal, on the surface of which is accurately engraven its divisions into degrees, etc. Many kinds of instruments used by surveyors, navigators, and astronomers, are constructed upon this principle.

210. An instrument called a protractor is used, in drawing, for measuring angles, and for laying down, on paper, angles of any required size. It consists of a semicircle of brass or mica, the circumference of which is divided into degrees and parts of a degree.

PROBLEMS IN DRAWING.

211. Problem.—*To draw an angle equal to a given angle.*

Let it be required to draw a line making, with the given line BC, an angle at B equal to the given angle A.



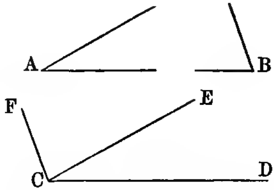
With A as a center, and any assumed radius AD, draw the arc DE cutting the sides of the angle A. With B as a center, and the same radius as before, draw an arc FG. Join DE. With F as a center, and a radius equal to DE, draw an arc cutting FG at the point G. Join BG. Then GBF is the required angle.

For, joining FG, the arcs DE and FG have equal radii and equal chords, and therefore are equal (185). Hence, they subtend equal angles (197).

212. Corollary.—An arc equal to a given arc may be drawn in the same way.

213. Problem.—*To draw an angle equal to the sum of two given angles.*

Let A and B be the given angles. First, make the angle DCE equal to A, and then at C, on the line CE, draw the angle ECF equal to B. The angle FCD is equal to the sum of A and B (9).



214. Corollary.—In a similar manner, draw an angle equal to the sum of several given angles; also, an angle equal to the difference of two given angles; or, an angle equal to the supplement, or to the complement of a given angle.

215. Corollary.—By the same methods, an arc may be drawn equal to the difference of two arcs having equal radii, or equal to the sum of several arcs.

216. Problem.—*To erect a perpendicular to a given line at its extreme point, without producing the given line.*

A right angle may be made separately, and then, at the end of the given line, an angle be made equal to the given angle.

This is the method universally employed by mechanics and draughtsmen to construct right angles and perpendiculars by the use of the *square.*

217. Problem.—*To draw a line through a given point parallel to a given line.*

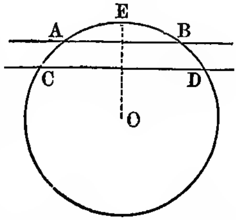
This has been done by means of perpendiculars (174). It may be done with an oblique secant, by making the alternate or the corresponding angles equal.

ARCS INTERCEPTED BY PARALLELS.

218. An arc which is included between two parallel lines, or between the sides of an angle, is called *intercepted.*

219. Theorem.—*Two parallel lines intercept equal arcs of a circumference.*

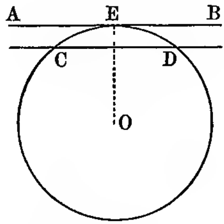
The two lines may be both secants, or both tangents, or one a secant and one a tangent.



1st. When both are secants.

The arcs AC and BD intercepted by the parallels AB and CD are equal.

For, let fall from the center O a perpendicular upon CD, and produce it to the circumference at E. Then OE is also perpendicular to AB (127). Therefore, the arcs EA and EB are equal (187); and the arcs EC and ED are equal. Subtracting the first from the second, there remains the arc AC equal to the arc BD.

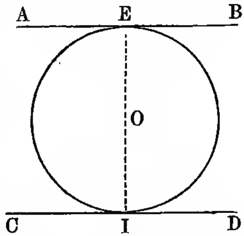


2d. When one is a tangent.

Extend the radius OE to the point of contact. This radius is perpendicular to the tangent AB (183). Hence, it is perpendicular to the secant CD (127), and therefore it bisects the arc CED at the point E (187). That is, the intercepted arcs EC and ED are equal.

3d. When both are tangents.

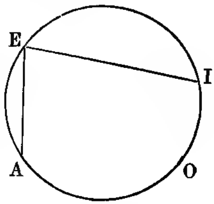
Extend the radii OE and OI to the points of contact. These radii being perpendicular (183) to the parallels, must (103 and 127) form one straight line. Therefore, EI is a diameter, and divides (157) the circumference into equal parts. But these equal parts are the arcs intercepted by the parallel tangents.



Therefore, in every case, the arcs intercepted by two parallels are equal.

ARCS INTERCEPTED BY ANGLES.

220. An INSCRIBED ANGLE is one whose sides are chords or secants, and whose vertex is on the circumference. An angle is said to be *inscribed in an arc*, when its vertex is on the arc and its sides extend to or through the ends of the arc. In such a case the arc is said *to contain the angle*. Thus, the angle AEI is inscribed in the arc AEI, and the arc AEI contains the angle AEI.



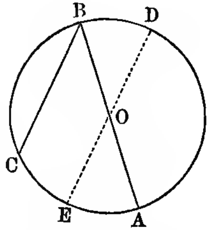
An angle is said to stand upon the arc intercepted between its sides. Thus, the angle AEI stands upon the arc AOI.

221. Corollary.—The arc in which an angle is inscribed, and the arc intercepted between its sides, compose the whole circumference.

222. Theorem.—*An inscribed angle is measured by half of the intercepted arc.*

This demonstration also presents three cases. The center of the circle may be on one of the sides of the angle, or it may be inside, or it may be outside of the angle.

1st. One side of the angle, as AB, may be a diameter.

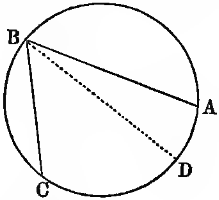


Make the diameter DE, parallel to BC, the other side of the angle. Then the angle B is equal to its alternate angle BOD (125), which is measured by the arc BD (207). This arc is equal to CE (219), and also to EA (197). Therefore, the arc

BD is equal to the half of AC, and the inscribed angle B is measured by half of its intercepted arc.

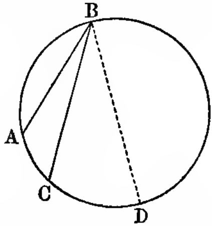
2d. The center of the circle may be within the angle.

From the vertex B extend a diameter to the opposite side of the circumference at D. As just proved, the angle ABD is measured by half of the arc AD, and the angle DBC by half of the arc DC. Therefore, the sum of the two angles, or ABC, is measured by half of the sum of the two arcs, or half of the arc ADC.



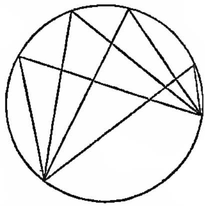
3d. The center of the circle may be outside of the angle.

Extend a diameter from the vertex as before. The angle ABC is equal to ABD diminished by DBC, and is, therefore, measured by half of the arc DA diminished by half of DC; that is, by the half of AC.



223. Corollary.—When an inscribed angle and an angle at the center have the same intercepted arc, the inscribed angle is half of the angle at the center.

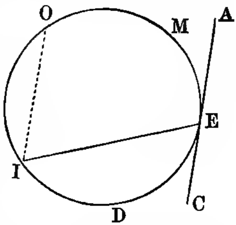
224. Corollary.—All angles inscribed in the same arc are equal, for they have the same measure.



225. Corollary.—Every angle inscribed in a semi-circumference is a right angle. If the arc is less than a semi-circumference, the angle is obtuse. If the arc is greater, the angle is acute.

226. Theorem.—*The angle formed by a tangent and a chord is measured by half the intercepted arc.*

The angle CEI, formed by the tangent AC and the chord EI, is measured by half the intercepted arc IDE.



Through I, make the chord IO parallel to the tangent AC.

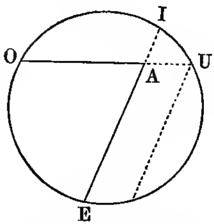
The angle CEI is equal to its alternate EIO (125), which is measured by half the arc OME (222), which is equal to the arc IDE (219). Therefore, the angle CEI is measured by half the arc IDE.

The sum of the angles AEI and CEI is two right angles, and is therefore measured by half the whole circumference (207). Hence, the angle AEI is equal to two right angles diminished by the angle CEI, and is measured by half the whole circumference diminished by half the arc IDE; that is, by half the arc IOME.

Thus it is proved that each of the angles formed at E, is measured by half the arc intercepted between its sides.

227. This theorem may be demonstrated very elegantly by the method of limits (200).

228. Theorem.—*Every angle whose vertex is within the circumference, is measured by half the sum of the arcs intercepted between its sides and its sides produced.*

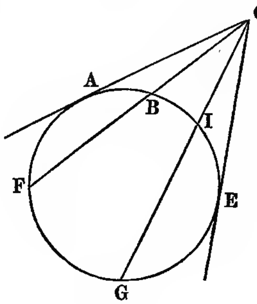


Thus, the angle OAE is measured by half the sum of the arcs OE and IU.

To be demonstrated by the student, using the previous theorems (219 and 222).

229. Theorem.—*Every angle whose vertex is outside of a circumference, and whose sides are either tangent or secant, is measured by half the difference of the intercepted arcs.*

Thus, the angle ACF is measured by half the difference of the arcs AF and AB; the angle FCG, by half the difference of the arcs FG and BI; and the angle ACE, by half the difference of the arcs AFGE and ABIE.

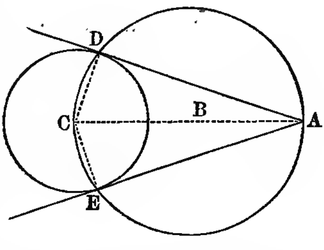


This, also, may be demonstrated by the student, by the aid of the previous theorems on intercepted arcs.

PROBLEMS IN DRAWING.

230. Problem.—*Through a given point out of a circumference, to draw a tangent to the circumference.*

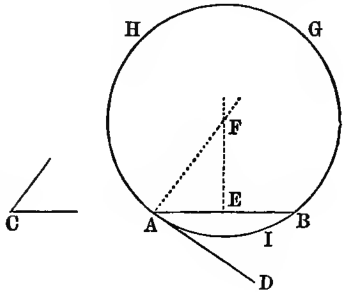
Let A be the given point, and C the center of the given circle. Join AC. Bisect AC at the point B (171). With B as a center and BC as a radius, describe a circumference. It will pass through C and A (153), and will cut the circumference in two points, D and E. Draw straight lines from A through D and E. AD and AE are both tangent to the given circumference.



Join CD and CE. The angle CDA is inscribed in a semicircumference, and is therefore a right angle (225). Since AD is perpendicular to the radius CD, it is tangent to the circumference (178). AE is tangent for the same reasons.

231. Problem.—*Upon a given chord to describe an arc which shall contain a given angle.*

Let AB be the chord, and C the angle. Make the angle DAB equal to C. At A erect a perpendicular to AD, and erect a perpendicular to AB at its center (172). Produce these till they meet at the point F (137). With F as a center, and FA as a radius, describe a circumference. Any angle inscribed in the arc BGHA will be equal to the given angle C.



For AD, being perpendicular to the radius FA, is a tangent (178). Therefore, the angle BAD is measured by half of the arc AIB (226). But any angle contained in the arc AHGB is also measured by half of the same arc (222), and is therefore equal to BAD, which was made equal to C.

POSITIONS OF TWO CIRCUMFERENCES.

232. Theorem.—*Two circumferences can not cut each other in more than two points.*

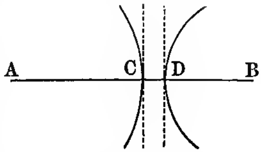
For three points determine the position and extent of a circumference (149). Therefore, if two circumferences have three points common, they must coincide throughout.

233. Let us investigate the various positions which two circumferences may have with reference to each other.

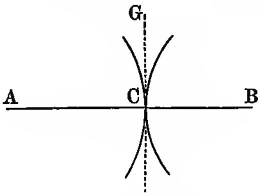
Let A and B be the centers of two circles, and let these points be joined by a straight line, which therefore measures the distance between the centers.

First, suppose the sum of the radii to be less than AB.

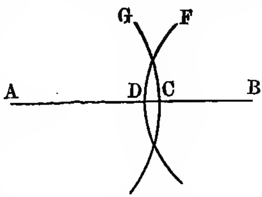
Then AC and BD, the radii, can not reach each other. At C and D, where the curves cut the line AB, let perpendiculars to that line be erected. These perpendiculars are parallel to each other (129). They are also tangent respectively to the two circumferences (178). It follows, therefore, that CD, the distance between these parallels, is also the least distance between the two curves.



234. Next, let the sum of the radii AC and BC be equal to AB, the distance between the centers. Then both curves will pass through the point C (153). At this point let a perpendicular be erected as before. This perpendicular CG is tangent to both the curves (178); that is, it is cut by neither of them. Therefore, the curves have only one common point C.

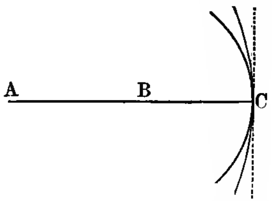


235. Next, let AB be less than the sum, but greater than the difference, of the radii AC and BD. Then the point C will fall within the circumference DF. For if it fell on or outside of it, on the side toward A, then AB would be equal to or greater than the sum of the radii; and if the point C fell on or outside of the curve in the direction toward B, then AB would be equal to or less than the difference between the radii. Each of these is contrary to the hypothesis. For the same reasons, the point D will fall within the



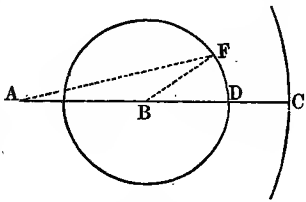
circumference CG. Therefore, these circumferences cut each other, and have two points common (232).

236. Next, let the difference between the two radii AC and BC be equal to the distance AB. A perpendicular to this line at the point C will be a tangent to both curves, and they have a common point at C. They have no other common point, for the two curves are both symmetrical about the line AC (158), and, therefore, if they had a common point on one side of that line, they would have a corresponding common point on the other side; but this can not be, for they would then have three points common (232).



237. Lastly, suppose the distance AB less than the difference of the radii AC and BD, by the line CD. That is,

$$AB + BD + DC = AC.$$



Join A, the center of the larger circle, with F, any point of the smaller circumference, and join BF. Then AB and BD are together equal to AB and BF, which are together greater than AF. Therefore, AD is greater than AF. Hence, the point D is farther from A than any other point of the circumference DF. It follows that CD is the least distance between the two curves.

The above course of reasoning develops the following principles:

238. Theorem.—*Two circumferences may have, with reference to each other, five positions:*

1*st. Each may be entirely exterior to the other, when the distance between their centers is greater than the sum of their radii.*

2*d. They may touch each other exteriorly, having one point common, when the distance between the centers is equal to the sum of the radii.*

3*d. They may cut each other, having two points common, when the distance between the centers is less than the sum and greater than the difference of the radii.*

4*th. One may be within the other and tangent, having one point common, when the distance between the centers is equal to the difference of the radii.*

5*th. One may be entirely within the other, when the distance between the centers is less than the difference of the radii.*

239. Corollary.—Two circumferences can not have more than one chord common to both.

240. Corollary—The common chord of two circumferences is perpendicular to the straight line which joins their centers and is bisected by it. For the ends of the chords are equidistant from each of the centers, the ends of the other line (109).

241. Corollary.—When two circumferences are tangent to each other, the two centers and the point of contact are in one straight line.

242. Corollary.—When two circumferences have no common point, the least distance between the curves is measured along the line which joins the centers.

243. Corollary.—When the distance between the centers is zero, that is, when they coincide, a straight line through this point may have any direction in the plane; and the two curves are equidistant at all points. Such circles are called CONCENTRIC.

244. A Locus is a line or a surface all the points of which have some common property, which does not belong to any other points. This is also frequently called a *geometrical locus*. Thus,

The circumference of a circle is the locus of all those points in the plane, which are at a given distance from a given point.

A straight line perpendicular to another at its center is the locus of all those points in the plane, which are at the same distance from both ends of the second line.

The geometrical locus of the centers of those circles which have a given radius, and are tangent to a given straight line, is a line parallel to the former, and at a distance from it equal to the radius.

245. The student will find an excellent review of the preceding pages, in demonstrating the theorems, and solving the problems in drawing which follow.

In his efforts to discover the solutions of the more difficult problems in drawing, the student will be much assisted by the following

SUGGESTIONS.—1. Suppose the problem solved, and the figure completed.

2. Find the geometrical relations of the different parts of the figure thus formed, drawing auxiliary lines, if necessary.

3. From the principles thus developed, make a rule for the solution of a problem.

This is the *analytic* method of solving problems.

EXERCISES.

1. Take two straight lines at random, and find their ratio.

Make examples in this way for all the problems in drawing.

2. Bisect a quadrant, also its half, its fourth; and so on.

3. From a given point, to draw the shortest line possible to a given straight line.

4. With a given length of radius, to draw a circumference through two given points.

5. From two given points, to draw two equal straight lines which shall end in the same point of a given line.

6. From a point out of a straight line, to draw a second line making a required angle with the first.

7. If from a point without a circle two straight lines extend to the concave part of the circumference, making equal angles with the line joining the same point and the center of the circle, then the parts of the first two lines which are within the circumference are equal.

8. To draw a line through a point such that the perpendiculars upon this line, from two other points, may be equal.

9. From two points on the same side of a straight line, to draw two other straight lines which shall meet in the first, and make equal angles with it.

10. In each of the five cases of the last theorem (238), how many straight lines can be tangent to both circumferences?

The number is different for each case.

11. On any two circumferences, the two points which are at the greatest distance apart are in the prolongation of the line which joins the centers.

12. To draw a circumference with a given radius, through a given point, and tangent to a given straight line.

13. With a given radius, to draw a circumference tangent to two given circumferences.

14. What is the locus of the centers of those circles which have a given radius, and are tangent to a given circle?

15. Of all straight lines which can be drawn from two given points to meet on the convex circumference of a circle, the sum of those two is the least which make equal angles with the tangent to the circle at the point of concourse.

16. If two circumferences be such that the radius of one is the diameter of the other, any straight line extending from their point of contact to the outer circumference is bisected by the inner one.

17. If two circumferences cut each other, and from either point of intersection a diameter be made in each, the extremities of these diameters and the other point of intersection are in the same straight line.

18. If any straight line joining two parallel lines be bisected, any other line through the point of bisection and joining the two parallels, is also bisected at that point.

19. If two circumferences are concentric, a line which is a chord of the one and a tangent of the other, is bisected at the point of contact.

20. If a circle have any number of equal chords, what is the locus of their points of bisection?

21. If any point, not the center, be taken in a diameter of a circle, of all the chords which can pass through that point, that one is the least which is at right angles to the diameter.

22. If from any point there extend two lines tangent to a circumference, the angle contained by the tangents is double the angle contained by the line joining the points of contact and the radius extending to one of them.

23. If from the ends of a diameter perpendiculars be let fall on any line cutting the circumference, the parts intercepted between those perpendiculars and the curve are equal.

24. To draw a circumference with a given radius, so that the sides of a given angle shall be tangents to it.

25. To draw a circumference through two given points, with the center in a given line.

26. Through a given point, to draw a straight line, making equal angles with the two sides of a given angle.

CHAPTER V.

TRIANGLES.

246. Next in regular order is the consideration of those plane figures which inclose an area; and, first, of those whose boundaries are straight lines.

A Polygon is a portion of a plane bounded by straight lines. The straight lines are the *sides* of the polygon.

The Perimeter of a polygon is its boundary, or the sum of all the sides. Sometimes this word is used to designate the boundary of any plane figure.

247. A Triangle is a polygon of three sides.

Less than three straight lines can not inclose a surface, for two straight lines can have only one common point (51). Therefore, the triangle is the simplest polygon. From a consideration of its properties, those of all other polygons may be derived.

248. Problem.—*Any three points not in the same straight line may be made the vertices of the three angles of a triangle.*

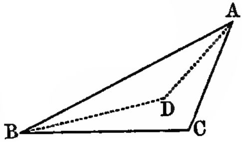
For these points determine the plane (60), and straight lines may join them two and two (47), thus forming the required figure.

INSCRIBED AND CIRCUMSCRIBED.

249. Corollary.—Any three points of a circumference may be made the vertices of a triangle. A circumference may pass through the vertices of any triangle, for it may pass through any three points not in the same straight line (149).

250. Theorem.—*Within every triangle there is a point equally distant from the three sides.*

In the triangle ABC, let lines bisecting the angles A and B be produced until they meet.

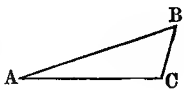


The point D, where the two bisecting lines meet, is equally distant from the two sides AB and BC, since it is a point of the line which bisects the angle B (113). For a similar reason, the point D is equally distant from the two sides AB and AC. Therefore, it is equally distant from the three sides of the triangle.

251. Corollary.—The three lines which bisect the several angles of a triangle meet at one point. For the point D must be in the line which bisects the angle C (113).

252. Corollary.—With D as a center, and a radius equal to the distance of D from either side, a circumference may be described, to which every side of the triangle will be a tangent.

253. When a circumference passes through the vertices of all the angles of a polygon, the circle is said to be *circumscribed* about the polygon, and the polygon to be *inscribed* in the circle. When every side of a polygon is tangent to a circumference, the circle is *inscribed* and the polygon *circumscribed.*

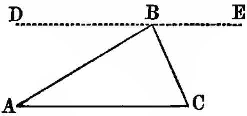


254. The angles at the ends of one side of a triangle are said to be *adjacent* to that side. Thus, the angles A and B are adjacent to the side AB. The angle formed by the other two sides is *opposite.* Thus, the angle A and the side BC are opposite to each other.

SUM OF THE ANGLES.

255. Theorem.—*The sum of the angles of a triangle is equal to two right angles.*

Let the line DE pass through the vertex of one angle, B, parallel to the opposite side, AC.



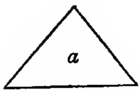
Then the angle A is equal to its alternate angle DBA (125). For the same reason, the angle C is equal to the angle EBC. Hence, the three angles of the triangle are equal to the three consecutive angles at the point B, which are equal to two right angles (91). Therefore, the sum of the three angles of the triangle is equal to two right angles.

256. Corollary.—Each angle of a triangle is the supplement of the sum of the other two.

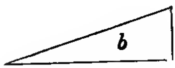
257. Corollary.—At least two of the angles of a triangle are acute.

258. Corollary.—If two angles of a triangle are equal, they are both acute. If the three are equal, they are all acute, and each is two-thirds of a right angle.

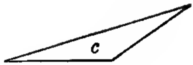
259. An ACUTE ANGLED triangle is one which has all its angles acute, as *a*.



A RIGHT ANGLED triangle has one of the angles right, as *b*.

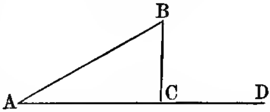


An OBTUSE ANGLED triangle has one of the angles obtuse, as *c*.



260. Corollary.—In a right angled triangle, the two acute angles are complementary (94).

261. Corollary.—If one side of a triangle be produced, the exterior angle thus formed, as BCD, is equal to the sum of the two interior angles not adjacent to it, as A and B (256). So much the more, the exterior angle is greater than either one of the interior angles not adjacent to it.



262. Corollary.—If two angles of a triangle are respectively equal to two angles of another, then the third angles are also equal.

263. Either side of a triangle may be taken as the *base*. Then the vertex of the angle opposite the base is the *vertex* of the triangle.

The ALTITUDE of the triangle is the distance from the vertex to the base, which is measured by a perpendicular let fall on the base produced, if necessary.

264. Corollary.—The altitude of a triangle is equal to the distance between the base and a line through the vertex parallel to the base.

265. When one of the angles at the base is obtuse, the perpendicular falls outside of the triangle.

When one of the angles at the base is right, the altitude coincides with the perpendicular side.

When both the angles at the base are acute, the altitude falls within the triangle.

Let the student give the reason for each case, and illustrate it with a diagram.

LIMITS OF SIDES.

266. Theorem.—*Each side of a triangle is smaller than the sum of the other two, and greater than their difference.*

The first part of this theorem is an immediate consequence of the Axiom of Distance (54); that is,

$$AC < AB + BC.$$

Subtract AB from both members of this inequality, and

$$AC - AB < BC.$$

That is, BC is greater than the difference of the other sides.

Prove the same for each of the other sides.

267. An Equilateral triangle is one which has three sides equal.

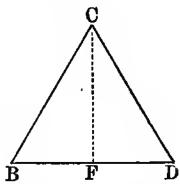
An Isosceles triangle is one which has only two sides equal.

A Scalene triangle is one which has no two sides equal.

EQUAL SIDES.

268. Theorem.—*When two sides of a triangle are equal, the angles opposite to them are equal.*

If the triangle BCD is isosceles, the angles B and D, which are opposite the equal sides, are equal.



Let the angle C be divided into two equal parts, and let the dividing line extend to the opposite side of the triangle at F.

Then, that portion of the figure upon one side of this line may be turned upon it as upon an axis. Since the angle C was bisected, the line BC will fall upon DC; and, since these two lines are equal, the point B will fall upon D. But F, being a point of the axis, remains fixed; hence, BF and DF will coincide. Therefore, the angles B and D coincide, and are equal.

269. Corollary.—The three angles of an equilateral triangle are equal.

270. In an isosceles triangle, the angle included by the equal sides is usually called the *vertex* of the triangle, and the side opposite to it the *base*.

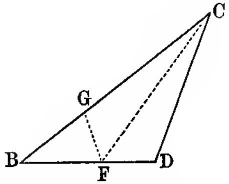
271. Corollary.—If a line pass through the vertex of an isosceles triangle, and also through the middle of the base, it will bisect the angle at the vertex, and be perpendicular to the base.

The straight line which has any two of these four conditions must have the other two (52).

UNEQUAL SIDES.

272. Theorem.—*When two sides of a triangle are unequal, the angle opposite to the greater side is greater than the angle opposite to the less side.*

If in the triangle BCD the side BC is greater than DC, then the angle D is greater than the angle B.



Let the line CF bisect the angle C, and be produced to the side BD. Then let the triangle CDF turn upon CF. CD will take the direction CB; but, since CD is less than CB, the point D will fall between C and B, at G. Join GF.

Now, the angle FGC is equal to the angle D, because they coincide; and it is greater than the angle B, because it is exterior to the triangle BGF (261). Therefore, the angle D is greater than B.

273. Corollary.—When one side of a triangle is not the largest, the angle which is opposite to that side is acute (257).

274. Corollary.—In a scalene triangle, no two angles are equal.

EQUAL ANGLES.

275. Theorem.—*If two angles of a triangle are equal, the sides opposite them are equal.*

For if these sides were unequal, the angles opposite to them would be unequal (272), which is contrary to the hypothesis.

276. Corollary.—If a triangle is *equiangular*, that is, has all its angles equal, then it is equilateral.

UNEQUAL ANGLES.

277. Theorem.—*If two angles of a triangle are unequal, the side opposite to the greater angle is greater than the side opposite to the less.*

If, in the triangle ABC, the angle C is greater than the angle A, then AB is greater than BC.

For, if AB were not greater than BC, it would be either equal to it or less. If AB were equal to BC, the opposite angles A and C would be equal (268); and if AB were less than BC, then the angle C would be less than A (272); but both of these conclusions are contrary to the hypothesis. Therefore, AB being neither less than nor equal to BC, must be greater.

278. Corollary.—In an obtuse angled triangle, the longest side is opposite the obtuse angle; and in a right angled triangle, the longest side is opposite the right angle.

279. The HYPOTENUSE of a right angled triangle is the side opposite the right angle. The other two sides are called the *legs*.

The student will notice that some of the above propositions are but different statements of the principles of perpendicular and oblique lines.

EXERCISES.

280.—1. How many degrees are there in an angle of an equilateral triangle?

2. If one of the angles at the base of an isosceles triangle be double the angle at the vertex, how many degrees in each?

3. If the angle at the vertex of an isosceles triangle be double one of the angles at the base, what is the angle at the vertex?

4. To circumscribe a circle about a given triangle (149).

5. To inscribe a circle in a given triangle (252).

6. If two sides of a triangle be produced, the lines which bisect the two exterior angles and the third interior angle all meet in one point.

7. Draw a line DE parallel to the base BC of a triangle ABC, so that DE shall be equal to the sum of BD and CE.

8. Can a triangular field have one side 436 yards, the second 547 yards, and the third 984 yards long?

9. The angle at the base of an isosceles triangle being one-fourth of the angle at the vertex, if a perpendicular be erected to the base at its extreme point, and this perpendicular meet the opposite side of the triangle produced, then the part produced, the remaining side, and the perpendicular form an equilateral triangle.

10. If with the vertex of an isosceles triangle as a center, a circumference be drawn cutting the base or the base produced, then the parts intercepted between the curve and the extremities of the base, are equal.

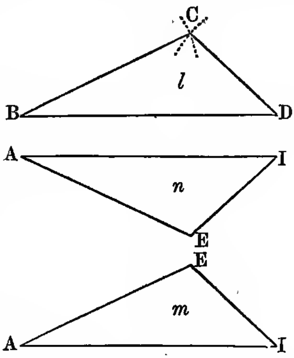
EQUALITY OF TRIANGLES.

281. The three sides and three angles of a triangle may be called its *six elements*. It may be shown that three of these are always necessary, and they are generally enough, to determine the triangle.

THREE SIDES EQUAL.

282. Theorem.—*Two triangles are equal when the three sides of the one are respectively equal to the three sides of the other.*

Let the side BD be equal to AI, the side BC equal to AE, and CD to EI; then the two triangles are equal.



Apply the line AI to its equal BD, so that the point A will fall upon B. Then I will fall upon D, since the lines are equal. Next, turn one of the triangles, if necessary, so that both shall fall on the same side of this common line.

Now, the point A being on B, the points E and C are at the same distance from B, and therefore they are both in the circumference, which has B for its center, and BC or AE for its radius (153). For a similar reason, the points E and C are both in the circumference, which has D for its center and DC or IE for its radius. These two circumferences have only one point common on one side of the line BD, which joins their centers (232). Hence, E and C are both at this point. Therefore (51), AE coincides with BC, and EI with CD; that is, the two triangles coincide throughout, and are equal.

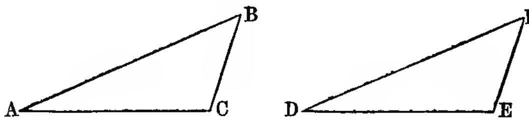
283. Every plane figure may be supposed to have two *faces*, which may be termed the upward and the downward faces. In order to place the triangle *m* upon *l*, we may conceive it to slide along the plane without turning over; but, in order to place *n* upon *l*, it must be turned over, so that its upward face will be upon the upward face of *l*.

There are, then, two methods of superposition; the first, called *direct*, when the downward face of one figure is applied to the upward face of the other; and the second, called *inverse*, when the upward faces of the two are applied to each other. Hitherto, we have used only the inverse method. Generally, in the chapter on the circumference, either method might be used indifferently.

TWO SIDES AND INCLUDED ANGLE.

284. Theorem.—*Two triangles are equal when they have two sides and the included angle of the one, respectively equal to two sides and the included angle of the other.*

If the angle A is equal to D, and the side AB to



the side DF, and AC to DE, then the two triangles are equal.

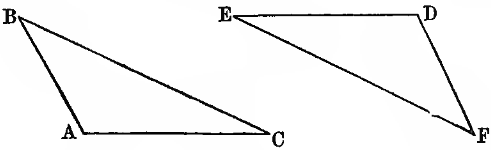
Apply the side AC to its equal DE, turning one triangle, if necessary, so that both shall fall upon the same side of that common line.

Then, since the angles A and D are equal, AB must take the direction DF, and these lines being equal, B will fall upon F. Therefore, BC and FE, having two points common, coincide; and the two triangles coincide throughout, and are equal.

ONE SIDE AND TWO ANGLES.

285. Theorem.—*Two triangles are equal when they have one side and two adjacent angles of the one, respectively equal to a side and the two adjacent angles of the other.*

If the triangles ABC and DEF have the side AC



equal to DE, and the angle A equal to D, and C equal to E, then the triangles are equal.

Apply the side AC to its equal DE, so that the vertices of the equal angles shall come together, A upon D, and C upon E, and turning one triangle, if necessary, so that both shall fall upon one side of the common line.

Then, since the angles A and D are equal, AB will take the direction DF, and the point B will fall somewhere in the line DF. Since the angles C and E are equal, CB will take the direction EF, and B will also be in the line EF. Therefore, B falls upon F, the only point common to the two lines DF and EF. Hence, the sides of the one triangle coincide with those of the other, and the two triangles are equal.

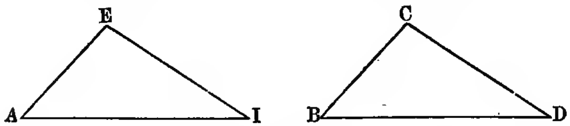
286. Theorem.—*Two triangles are equal when they have one side and any two angles of the one, respectively equal to the corresponding parts of the other.*

For the third angle of the first triangle must be equal to the third angle of the other (262). Then this becomes a case of the preceding theorem.

TWO SIDES AND AN OPPOSITE ANGLE.

287. Theorem.—*Two triangles are equal when one of them has two sides, and the angle opposite to the side which is equal to or greater than the other, respectively equal to the corresponding parts of the other triangle.*

Let the sides AE and EI, EI being equal to or



greater than AE, and the angle A, be respectively equal to the sides BC, CD, and the angle B. Then the triangles are equal.

For the side AE may be placed on its equal BC. Then, since the angles A and B are equal, AI will take the direction BD, and the points I and D will both be in the common line BD. Since EI and CD are equal, the points I and D are both in the circumference whose center is at C, and whose radius is equal to CD. Now, this circumference cuts a straight line extending from B toward D in only one point; for B is either within or on the circumference, since BC is equal to or less than CD. Therefore, I and D are both at that point.

Hence, AI and BD are equal, and the triangles are equal (282).

288. Corollary.—Two triangles are equal when they have an obtuse or a right angle in the one, together with the side opposite to it, and one other side, respectively equal to those parts in the other triangle (278).

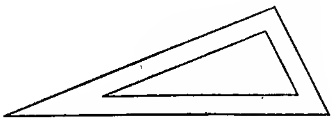
The two following are corollaries of the last five theorems, and of the definition (40).

289. Corollary.—In equal triangles each part of one is equal to the corresponding part of the other.

290. Corollary.—In equal triangles the equal parts are similarly arranged, so that equal angles are opposite to equal sides.

EXCEPTIONS TO THE GENERAL RULE.

291. A general rule as to the equality of triangles has been given (281). There are two exceptions.

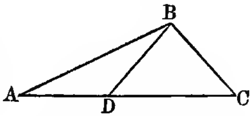


1. When the three angles are given.

For two very unequal triangles may have the angles of one equal to those of the other.

2. When two unequal sides and the angle opposite to the less are given.

For with the sides AB and BC and the angle A given, there are two triangles, ABC and ABD.

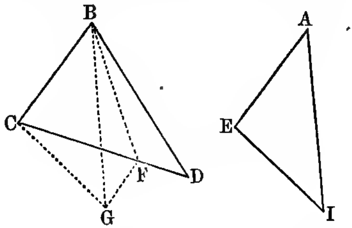


292. The student may show that two parts alone are never enough to determine a triangle.

UNEQUAL TRIANGLES.

293. Theorem.—*When two triangles have two sides of the one respectively equal to two sides of the other, and the included angles unequal, the third side in that triangle which has the greater angle, is greater than in the other.*

Let BCD and AEI be two triangles, having BC equal to AE, and BD equal to AI, and the angle A less than B. Then, it is to be proved that CD is greater than EI.



Apply the triangle AEI to BCD, so that AE will coincide with its equal BC. Since the angle A is less than B, the side AI will fall within the angle CBD. Let BG be its position, and EI will fall upon CG. Then let a line BF bisect the angle GBD. Join FG.

The triangles GBF and BDF have the side BF common, the side GB equal to the side DB, since each is equal to AI, and the included angles GBF and DBF equal by construction. Therefore, the triangles are equal (284), and FG is equal to FD (289). Hence, CD, the sum of CF and FD, is equal to the sum of CF and FG (7), which is greater than CG (54). Therefore, CD is greater than CG, or its equal EI.

If the point I should fall within the triangle BCD or on the line CD, the demonstration would not be changed.

294. Theorem.—*Conversely, if two triangles have two sides of the one equal to two sides of the other, and the third sides unequal, then the angles opposite the third sides are unequal, and that is greater which is opposite the greater side.*

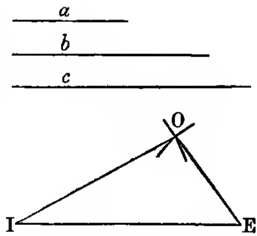
For if it were less, then the opposite side would be less (293), and if it were equal, then the opposite sides would be equal (284); both of which are contrary to the hypothesis.

PROBLEMS IN DRAWING.

295. Problem.—*To draw a triangle when the three sides are given.*

Let a, b, and c be the given lines.

Draw the line IE equal to c. With I as a center, and with the line b as a radius, describe an arc, and with E as a center and the line a as a radius, describe a second arc, so that the two may cut each other. Join O, the point of intersection of these arcs, with I and with E. IOE is the required triangle.



If c were greater than the sum of a and b, what would have been the result?

What, if c were less than the difference of a and b?

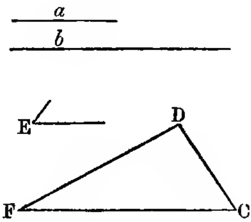
Has the problem more than one solution; that is, can unequal triangles be drawn which comply with the conditions? Why?

296. Corollary.—In the same way, draw a triangle equal to a given triangle.

297. Problem.—*To draw a triangle, two sides and the included angle being given.*

Let a and b be the given lines, and E the angle.

Draw FC equal to b. At C make an angle equal to E. Take DC equal to a, and join FD. Then FDC is a triangle having the required conditions.



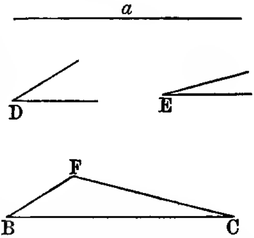
Has this problem more than one solution? Why?

Is this problem always soluble, whatever may be the size of the given angle, or the length of the given lines? Why?

298. Problem.—*To draw a triangle when one side and the adjacent angles are given.*

Let *a* be the given line, and D and E the angles.

Draw BC equal to *a*. At B make an angle equal to D, and at C an angle equal to E. Produce the sides till they meet at the point F. FBC is a triangle having the given side and angles.



Has this problem more than one solution?

Can it be solved, whatever be the given angles, or the given line?

299. Problem.—*To draw a triangle when one side and two angles are given.*

If one of the angles is opposite the given side, find the supplement of the sum of the given angles (214). This will be the other adjacent angle (256). Then proceed as in Article 298.

300. Problem.—*To draw a triangle when two sides and an angle opposite to one of them are given.*

Construct an angle equal to the given angle. Lay off on one side of the angle the length of the given adjacent side. With the extremity of this adjacent side as a center, and with a radius equal to the side opposite the given angle, draw an arc. This arc may cut the opposite side of the angle. Join the point of intersection with the end of the adjacent side which was taken as a center. A triangle thus formed has the required conditions.

The student can better discuss this problem after drawing several triangles with various given parts. Let the given angle vary from very obtuse to very acute; and let the opposite side vary from being much larger to much smaller than the side adjacent to the given angle. Then let the student explain when this problem has only one solution, when it has two, and when it can not be solved.

EXERCISES.

301.—1. The base of an isosceles triangle is to one of the other sides as three to two. Find by construction and measurement, whether the vertical angle is acute or obtuse.

2. Two right angled triangles are equal, when any two sides of the one are equal to the corresponding sides of the other.

3. Two right angled triangles are equal, when an acute angle and any side of the one are equal to the corresponding parts of the other.

4. Divide a given triangle into four equal parts.

5. Construct a right angled triangle when,

I. An acute angle and the adjacent leg are given;
II. An acute angle and the opposite leg are given;
III. A leg and the hypotenuse are given;
IV. When the two legs are given.

SIMILAR TRIANGLES.

302. Similar magnitudes have been defined to be those which have the same form while they differ in extent (37).

303. Let the student bear in mind that the form of a figure depends upon the relative directions of its points, and that angles are differences in direction. Therefore, the definition may be stated thus:

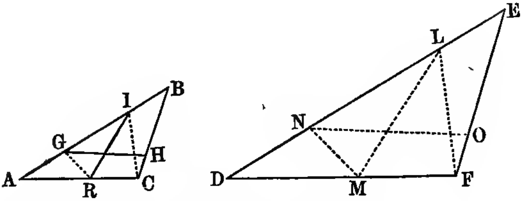
Two figures are similar when every angle that can be formed by lines joining points of one, has its corresponding equal and similarly situated angle in the other.

ANGLES EQUAL.

304. Theorem.—*Two triangles are similar, when the three angles of the one are respectively equal to the three angles of the other.*

This may appear to be only a case of the definition of similar figures; but it may be shown that every angle that can be made by any lines whatever in the one, may have its corresponding equal and similarly situated angle in the other.

Let the angles A, B, and C be respectively equal to



the angles D, E, and F. Suppose GH and IR to be any two lines in the triangle ABC.

Join IC and GR. From F, the point homologous to C, extend FL, making the angle LFE equal to ICB.

Now, the triangles LFE and ICB have the angles B and E equal, by hypothesis, and the angles at C and F equal, by construction. Therefore, the third angles, ELF and BIC, are equal (262). By subtraction, the angles AIC and DLF are equal, and the angles ACI and DFL.

From L extend LM, making the angle FLM equal to CIR. Then the two triangles FLM and CIR have the angles at C and F equal, as just proved, and the angles at I and L equal, by construction. Therefore, the third angles, LMF and IRC, are equal.

Join RG. Construct MN homologous to RG, and NO homologous to GH. Then show, by reasoning in the same manner, that the angles at N are equal to the corresponding angles at G; and so on, throughout the two figures.

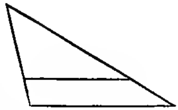
The demonstration is similar, whatever lines be first made in one of the triangles.

Therefore, the relative directions of all their points are the same in both triangles; that is, they have the same form. Therefore, they are similar figures.

305. Corollary.—Two similar triangles may be divided into the same number of triangles respectively similar, and similarly arranged.

306. Corollary.—Two triangles are similar, when two angles of the one are respectively equal to two angles of the other. For the third angles must be equal also (262).

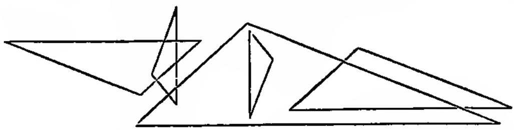
307. Corollary.—If two sides of a triangle be cut by a line parallel to the third side, the triangle cut off is similar to the original triangle (124).



308. Theorem.—*Two triangles are similar, when the sides of one are parallel to those of the other; or, when the sides of one are perpendicular to those of the other.*

We know (138 and 139) that the angles formed by lines which are parallel are either equal or supplementary; and that the same is true of angles whose sides are perpendicular (140). We will show that the angles can not be supplementary in two triangles.

If even two angles of one triangle could be respect-

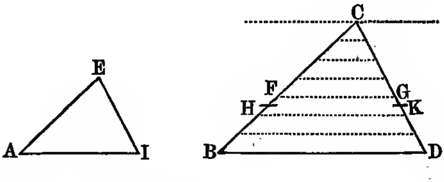


ively supplementary to two angles of another, the sum of these four angles would be four right angles; and then the sum of all the angles of the two triangles would be more than four right angles, which is impossible (255). Hence, when two triangles have their sides respectively parallel or perpendicular, at least two of the angles of one triangle must be equal to two of the other. Therefore, the triangles are similar (306).

SIDES PROPORTIONAL.

309. Theorem.—*One side of a triangle is to the homologous side of a similar triangle as any side of the first is to the homologous side of the second.*

If AE and BC are homologous sides of similar tri-



angles, also EI and CD, then,

AE : BC : : EI : CD.

Take CF equal to EA, and CG equal to EI, and join FG. Then the triangles AEI and FCG are equal (284), and the angles CFG and CGF are respectively equal to the angles A and I, and therefore equal to the angles B and D. Hence, FG is parallel to BD (129). Let a line extend through C parallel to FG and BD.

Suppose BC divided at the point F into parts which have the ratio of two whole numbers, for example, four and three. Then let the line CF be divided into four, and BF into three equal parts. Let lines parallel to BD extend from the several points of division till they meet CD.

Since BC is divided into equal parts, the distances between these parallels are all equal (135). Therefore, CD is also divided into seven equal parts (134), of which CG has four. That is,

CF : CB : : CG : CD : : 4 : 7.

But if the lines BC and CF have not the ratio of two whole numbers, then let BC be divided into any number of equal parts, and a line parallel to BD pass through H, the point of division nearest to F. Such a line must divide CD and CB proportionally, as just proved; that is,

CH : CB :: CK : CD.

By increasing the number of the equal parts into which BC is divided, the points H and K may be made to approach within any conceivable distance of F and G. Therefore, CF and CG are the limits of those lines, CH and CK, which are commensurable with BC and CD; and we may substitute CF and CG in the last proportion for CH and CK.

Hence, whatever be the ratio of CF to CB, it is the same as that of CG to CD. By substituting for CF and CG the equal lines AE and EI, we have,

AE : BC :: EI : CD.

By similar reasoning it may be shown that

AI : BD :: EI : CD.

310. Corollary.—The ratio is the same between any two homologous lines of two similar triangles.

311. This ratio of any side of a triangle to the homologous side of a similar triangle, is called the *linear ratio* of the two figures.

312. Corollary.—The perimeters of similar triangles have the linear ratio of the two figures. For,

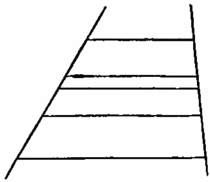
AE : BC :: EI : CD :: IA : DB.

Therefore (23),

AE+EI+IA : BC+CD+DB :: AE : BC.

313. Corollary.—If two sides of a triangle are cut by one or more lines parallel to the third side, the two sides are cut proportionally. For the triangles so formed are similar (307).

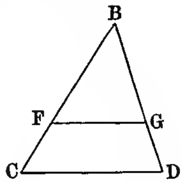
314. Corollary.—When several parallel lines are cut by two secants, the secants are divided proportionally.



For the secants being produced till they meet, form several similar triangles.

315. Theorem.—*If two sides of a triangle be cut proportionally by a straight line, the secant line is parallel to the third side.*

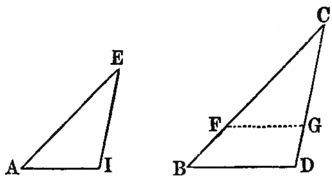
Let BCD be the triangle, and FG the secant.



A line parallel to CD may pass through F, and such a line must divide BD in the same ratio as BC (313). But, by hypothesis, BD is so divided at the point G. Therefore, a line through F parallel to CD, must pass through G, and coincide with FG. Hence, FG is parallel to CD.

316. Theorem.—*Two triangles are similar when the ratios between each side of the one and a corresponding side of the other are the same.*

Suppose AE : BC : : EI : CD : : AI : BD.



Take CF equal to EA and CG equal to EI, and join FG. Then,

CF : CB : : CG : CD.

Therefore, FG is parallel to BD (315), the triangles CFG and CBD are similar (307), and

CF : CB : : FG : BD.

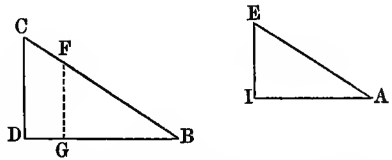
But, by hypothesis,

EA : CB :: AI : BD.

Hence, since CF is equal to EA, FG is equal to AI, and the triangles AEI and FCG are equal. Therefore, the triangles AEI and BCD have their angles equal, and are similar.

317. Theorem.—*Two triangles are similar when two sides of the one have respectively to two sides of the other the same ratio, and the included angles are equal.*

Suppose AE : BC :: AI : BD; and let the angle A be equal to B.



Take BF equal to AE, and BG equal to AI, and join FG. Then the triangles AEI and BFG are equal (284), and the angle BFG is equal to E, and BGF is equal to I. Since the sides of the triangle BCD are cut proportionally by FG, the angle BFG is equal to C, and BGF is equal to D (315). Therefore, the triangles AEI and BCD are mutually equiangular and similar.

318. If two similar triangles have two homologous lines equal, since all other homologous lines have the same ratio, they must also be equal, and consequently the two figures are equal. Thus, the equality of figures may be considered as a case of similarity.

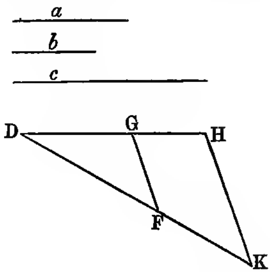
PROBLEMS IN DRAWING.

319. Problem.—*To find a fourth proportional to three given straight lines.*

Let a be the given extreme, and b and c the given means.

Take DG equal to a, the given extreme. Produce it, making

DH equal to c, one of the means. From G draw GF equal to b. Then, from D draw a line through F, and from H a line parallel to GF. Produce these two lines till they meet at the point K. HK is the required fourth proportional.



For the triangles DGF and DHK are similar (307). Hence,

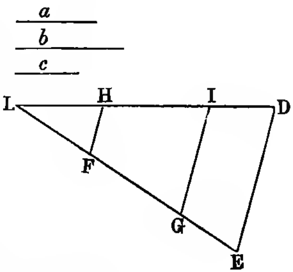
DG : GF :: DH : HK.

That is, $a : b :: c :$ HK.

It is most convenient to make GF and HK perpendicular to DH.

320. Problem.—*To divide a given line into parts having a certain ratio.*

Let LD be the line to be divided into parts proportional to the lines a, b, and c.



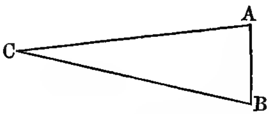
From L draw the line LE equal to the sum of a, b, and c, making LF equal to a, FG equal to b, and GE equal to c. Join DE, and draw GI and FH parallel to DE. LH, HI, and ID are the parts required.

The demonstration is similar to the last.

321. Problem.—*To divide a given line into any number of equal parts.*

This may be done by the last problem; but when the given line is small, the following method is preferable.

To divide the line AB into ten equal parts; draw AC indefinitely, and take on it ten equal parts. Join BC, and from the several points of division of AC, draw lines parallel to AB, and produce them to BC. The parallel nearest to AB is nine-tenths of AB, the next is eight-tenths, and so on.



This also depends upon similarity of triangles.

322. Problem.—*To draw a triangle on a given base, similar to a given triangle.*

Let this problem be solved by the student.

RIGHT ANGLED TRIANGLES.

323. Every triangle may be divided into two right angled triangles, by a perpendicular let fall from one of its vertices upon the opposite side. Thus the investigation of the properties of right angled triangles leads to many of the properties of triangles in general.

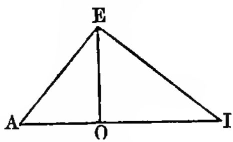
324. Theorem.—*If in a right angled triangle, a perpendicular be let fall from the vertex of the right angle upon the hypotenuse, then,*

1. *Each of the triangles thus formed is similar to the original triangle;*

2. *Either leg of the original triangle is a mean proportional between the hypotenuse and the adjacent segment of the hypotenuse; and,*

3. *The perpendicular is a mean proportional between the two segments of the hypotenuse.*

The triangles AEO and AEI have the angle A common, and the angles AEI and AOE are equal, being right angles. Therefore, these two triangles are similar (306)



That the triangles EOI and EIA are similar, is proved by the same reasoning.

Since the triangles are similar, the homologous sides are proportional, and we have

$$AI : AE :: AE : AO;$$

That is, the leg AE is a mean proportional between the whole hypotenuse and the segment AO which is adjacent to that leg.

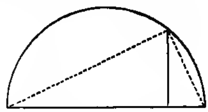
In like manner, prove that EI is a mean proportional between AI and OI.

Lastly, the triangles AEO and EIO are similar (304), and therefore,

$$AO : OE :: OE : OI.$$

That is, the perpendicular is a mean proportional between the two segments of the hypotenuse.

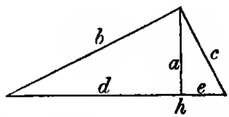
325. Corollary.—A perpendicular let fall from any point of a circumference upon a diameter, is a mean proportional between the two segments which it makes of the diameter. (225)



326. In the several proportions just demonstrated, in place of the lines we may substitute those numbers which constitute the ratios (14). Indeed, it is only upon this supposition that the proportions have a meaning. It is the same whether these numbers be integers or radicals, since we know that the terms of the ratio are in fact numbers.

327. Theorem.—*The second power of the length of the hypotenuse is equal to the sum of the second powers of the lengths of the two legs of a right angled triangle.*

Let h be the hypotenuse, a the perpendicular let fall upon it, b and c the legs, and d and e the corresponding segments of the hypotenuse made by the perpendicular. That is, these letters represent the number of times, whether integral or not, which some unit of length is contained in each of these lines.



By the second conclusion of the last theorem, we have

$$h : b :: b : d, \quad \text{and} \quad h : c :: c : e.$$

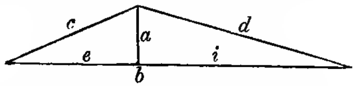
Hence, (16), $hd = b^2$, and $he = c^2$.

By adding these two, $h(d + e) = b^2 + c^2$.

But $d + e = h$. Therefore, $h^2 = b^2 + c^2$.

328. Theorem.—*If, in any triangle, a perpendicular be let fall from one of the vertices upon the opposite side as a base, then the whole base is to the sum of the other two sides, as the difference of those sides is to the difference of the segments of the base.*

Let a be the perpendicular, b the base, c and d the sides, and e and i the segments of the base.



Then, two right angled triangles are formed, in one of which we have

$$a^2 + i^2 = d^2;$$

and in the other, $a^2 + e^2 = c^2$.

Subtracting, $i^2 - e^2 = d^2 - c^2$.

Factoring, $(i + e)(i - e) = (d + c)(d - c)$.

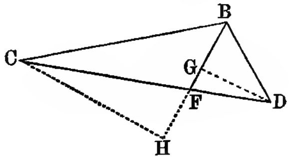
Whence (18), $i + e : d + c :: d - c : i - e$.

329. Theorem.—*If a line bisects an angle of a triangle, it divides the opposite side in the ratio of the adjacent sides.*

If BF bisects the angle CBD, then

$$\text{CF} : \text{FD} :: \text{CB} : \text{BD}.$$

This need be demonstrated only in the case where the sides adjacent to the bisected angle are not equal.



From C and from D, let perpendiculars DG and CH fall upon BF, and BF produced.

Then, the triangles BDG and BCH are similar, for they have equal angles at B, by hypothesis, and at G and H, by construction. Hence,

CB : BD :: CH : DG.

But the triangles DGF and CHF are also mutually equiangular and similar. Hence,

CF : FD :: CH : DG.

Therefore (21), CF · FD :: CB : BD.

330. Problem in Drawing.—*To find a mean proportional to two given straight lines.*

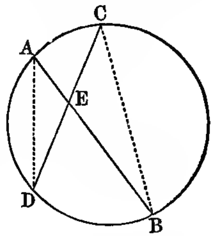
Make a straight line equal to the sum of the two. Upon this as a diameter, describe a semi-circumference. Upon this diameter, erect a perpendicular at the point of meeting of the two given lines. Produce this to the circumference. The line last drawn is the required line.

Let the student construct the figure and demonstrate.

CHORDS, SECANTS, AND TANGENTS.

331. Theorem.—*If two chords of a circle cut each other, the parts of one may be the extremes, and the parts of the other the means, of a proportion.*

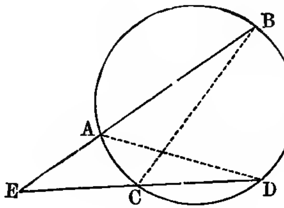
Join AD and CB. Then the two triangles AED and CEB have the angle A equal to the angle C, since they are inscribed in the same arc (224). For the same reason, the angles D and B are equal. Therefore, the triangles are similar (306); and we have (309),



AE : EC :: DE : EB.

332. Theorem.—*If from the same point, without a circle, two lines cutting the circumference extend to the farther side, then the whole of one secant and its exterior* *part may be the extremes, and the whole of the other secant and its exterior part may be the means, of a proportion.*

Joining BC and AD, the triangles AED and CEB are similar; for they have the angle E common, and the angles at B and D equal. Therefore,



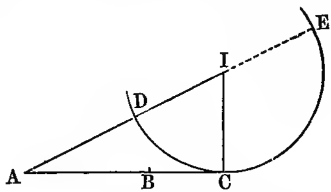
AE : EC :: DE : EB.

333. Corollary.—If from the same point there be a tangent and a secant, the tangent is a mean proportional between the secant and its exterior part. For the tangent is the limit of all the secants which pass through the point of meeting.

334. Problem in Drawing.—*To divide a given straight line into two parts so that one of them is a mean proportional between the whole line and the other part.*

This is called dividing a line in *extreme and mean ratio.*

Let AC be the given line. At C erect a perpendicular, CI, equal to half of AC. Join AI. Take ID equal to CI, and then AB equal to AD. The line AC is divided at the point B in extreme and mean ratio. That is,



AC : AB :: AB : BC.

With I as a center and IC as a radius, describe an arc DCE, and produce AI till it meets this arc at E. Then, AC is a tangent to this arc (178), and therefore (333),

AE : AC :: AC : AD.

Or (24), AE—AC : AC :: AC—AD : AD.

But AC is twice IC, by construction, and DE is twice IC, because DE is a diameter and IC is a radius. Therefore, the first term of the last proportion, AE — AC, is equal to AE — DE, which is AD; but AD is, by construction, equal to AB. Also, the third term, AC — AD, is equal to AC — AB, which is BC. And the fourth term is equal to AB. Substituting these equals, the proportion becomes

AB : AC :: BC : AB.

By inversion (19), AC : AB :: AB : BC.

ANALYSIS AND SYNTHESIS.

335. Geometrical Analysis is a process employed both for the discovery of the solution of problems and for the investigation of the truth of theorems. Analysis is the reverse of synthesis. Synthesis commences with certain principles, and proceeds by undeniable and successive inferences. The whole theory of geometry is an example of this method.

In the *analysis* of a problem, what was required to be done is supposed to have been effected, and the consequences are traced by a series of geometrical constructions and reasonings, till at length they terminate in the data of the problem, or in some admitted truth. See suggestions, Article 245.

In the *synthesis* of a problem, however, the last consequence of the analysis is the first step of the process, and the solution is effected by proceeding in a contrary order through the several steps of the analysis, until the process terminates in the thing required to be done.

If, in the analysis, we arrive at a consequence which conflicts with any established principle, or which is inconsistent with the data of the problem, then the solution is impossible. If, in certain relations of the given magnitudes, the construction is possible, while in other relations it is impossible, the discovery of these relations is a necessary part of the discussion of the problem.

In the analysis of a theorem, the question to be determined is, whether the proposition is true, as stated; and, if so, how this truth is to be demonstrated. To do this, the truth is assumed, and the successive consequences of this assumption are deduced till they terminate in the hypothesis of the theorem, or in some established truth.

The theorem will be proved synthetically by retracing, in order, the steps of the investigation pursued in the analysis, till they terminate in the conclusion which had been before assumed. This constitutes the demonstration.

If, in the analysis, the assumption of the truth of the proposition leads to some consequence which conflicts with an established principle, the false conclusion thus arrived at indicates the falsehood of the proposition which was assumed to be true.

In a word, analysis is used in geometry in order to discover truths, and synthesis to demonstrate the truths discovered.

Most of the problems and theorems which have been given for EXERCISES, are of so simple a character as scarcely to admit of the principle of geometrical analysis being applied to their solution.

336. A problem is said to be *determinate* when it admits of one definite solution; but when the same construction may be made on the other side of any given line, it is not considered a different solution. A problem is *indeterminate* when it admits of more than one definite solution. Thus, Article 300 presents a case where the problem may be determinate, indeterminate, or insolvable, according to the size of the given angle and extent of the given lines.

The solution of an indeterminate problem frequently amounts to finding a geometrical locus; as, to find a point equidistant from two given points; or, to find a point at a given distance from a given line.

EXERCISES.

337. Nearly all the following exercises depend upon principles found in this chapter, but a few of them depend on those of previous chapters.

1. If there be an isosceles and an equilateral triangle on the same base, and if the vertex of the inner triangle is equally distant from the vertex of the outer one and from the ends of the base, then, according as the isosceles triangle is the inner or the outer one, its base angle will be $\frac{1}{4}$ of, or $2\frac{1}{2}$ times the vertical angle.

2. The semi-perimeter of a triangle is greater than any one of the sides, and less than the sum of any two.

3. Through a given point, draw a line such that the parts of it, between the given point and perpendiculars let fall on it from two other given points, shall be equal.

What would be the result, if the first point were in the straight line joining the other two?

4. Of all triangles on the same base, and having their vertices in the same line parallel to the base, the isosceles has the greatest vertical angle.

5. If, from a point without a circle, two tangents be made to the circle, and if a third tangent be made at any point of the circumference between the first two, then, at whatever point the last tangent be made, the perimeter of the triangle formed by these tangents is a constant quantity.

6. Through a given point between two given lines, to draw a line such that the part intercepted by the given lines shall be bisected at the given point.

7. From a point without two given lines, to draw a line such that the part intercepted between the given lines shall be equal to the part between the given point and the nearest line.

8. The middle point of a hypotenuse is equally distant from the three vertices of a right angled triangle.

9. Given one angle, a side adjacent to it, and the difference of the other two sides, to construct the triangle.

10. Given one angle, a side opposite to it, and the difference of the other two sides, to construct the triangle.

11. Given one angle, a side opposite to it, and the sum of the other two sides, to construct the triangle.

12. Trisect a right angle.

13. If a circle be inscribed in a right angled triangle, the difference between the hypotenuse and the sum of the two legs is equal to the diameter of the circle.

14. If from a point within an equilateral triangle, a perpendicular line fall on each side, the sum of these perpendiculars is a constant quantity.

How should this theorem be stated, if the point were outside of the triangle?

15. Find the locus of the points such that the sum of the distances of each from the two sides of a given angle, is equal to a given line.

16. Find the locus of the points such that the difference of the distances of each from two sides of a given angle, is equal to a given line.

17. Demonstrate the last corollary (333) by means of similar triangles.

18. To draw a tangent common to two given circles.

19. To construct an isosceles triangle, when one side and one angle are given.

20. If in a right angled triangle one of the acute angles is equal to twice the other, then the hypotenuse is equal to twice the shorter leg.

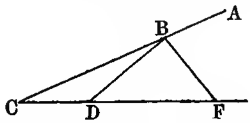
21. Draw a line DE parallel to the base BC of a triangle ABC, so that DE shall be equal to the difference of BD and CE.

22. In a given circle, to inscribe a triangle similar to a given triangle.

23. In a given circle, find the locus of the middle points of those chords which pass through a given point.

24. To describe a circumference tangent to three given equal circumferences, which are tangent to each other.

25. If a line bisects an exterior angle of a triangle, it divides the base produced into segments which are proportional to the adjacent sides. That is, if BF bisects the angle ABD, then,



CF : FD :: CB : BD.

26. The parts of two parallel lines intercepted by several straight lines which meet at one point, are proportional.

The converging lines are also divided in the same ratio.

27. Two triangles are similar, when two sides of one are proportional to two sides of the other, and the angle opposite to that side which is equal to or greater than the other given side in one, is equal to the homologous angle in the other.

28. The perpendiculars erected upon the several sides of a triangle at their centers, meet in one point.

29. The lines which bisect the several angles of a triangle, meet in one point.

30. The altitudes of a triangle, that is, the perpendiculars let fall from the several vertices on the opposite sides, meet in one point.

31. The lines which join the several vertices of a triangle with the centers of the opposite sides, meet in one point.

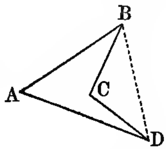
32. Each of the lines last mentioned is divided at the point of meeting into two parts, one of which is twice as long as the other.

CHAPTER VI.

QUADRILATERALS.

338. In a polygon, two angles which immediately succeed each other in going round the figure, are called *adjacent* angles. The student will distinguish adjacent angles of a polygon from the adjacent angles defined in Article 85.

A Diagonal of a polygon is a straight line joining the vertices of any two angles which are not adjacent. Sometimes a diagonal is exterior, as the diagonal BD of the figure ABCD.



A Convex polygon has all its diagonals interior.

A Concave polygon has at least one diagonal exterior, as in the above diagram.

Angles, such as BCD, are called *reëntrant.*

339. A Quadrilateral is a polygon of four sides.

340. Corollary.—Every quadrilateral has two diagonals.

341. Corollary.—An interior diagonal of a quadrilateral divides the figure into two triangles.

EQUAL QUADRILATERALS.

342. Theorem.—*Two quadrilaterals are equal when they are each composed of two triangles, which are respectively equal, and similarly arranged.*

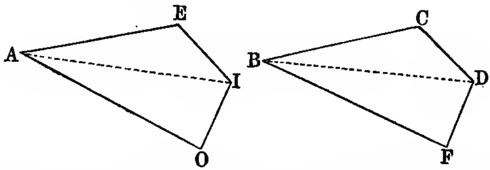
For, since the parts are equal and similarly arranged, the wholes may be made to coincide (40).

343. Corollary.—Conversely, two equal quadrilaterals may be divided into equal triangles similarly arranged. In every convex quadrilateral this division may be made in either of two ways.

344. Theorem.—*Two quadrilaterals are equal when the four sides and a diagonal of one are respectively equal to the four sides and the same diagonal of the other.*

By the same diagonal is meant the diagonal that has the same position with reference to the equal sides.

For, since all their sides are equal, the triangles AEI

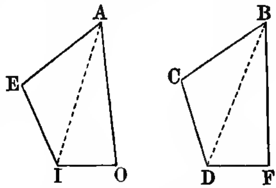


and BCD are equal, also the triangles AIO and BDF (282). Therefore, the quadrilaterals are equal (342).

345. Theorem.—*Two quadrilaterals are equal when the four sides and an angle of the one are respectively equal to the four sides and the similarly situated angle of the other.*

By the similarly situated angle is meant the angle included by equal sides.

For, if the sides AE, IE, and the included angle E are respectively equal to the side BC, DC, and the included angle C, then the triangles AEI and BCD are equal (284); and AI is equal to BD. But since the



three sides of the triangles AIO and BDF are respectively equal, the triangles are equal (282). Hence, the quadrilaterals are equal (342).

SUM OF THE ANGLES.

346. Theorem.—*The sum of the angles of a quadrilateral is equal to four right angles.*

For the angles of the two triangles into which every quadrilateral may be divided, are together coincident with the angles of the quadrilateral. Therefore, the sum of the angles of a quadrilateral is twice the sum of the angles of a triangle.

Let the student illustrate this with a diagram.

In applying this theorem to a concave figure (338), the value of the reëntrant angle must be taken on the side toward the polygon, and therefore as amounting to more than two right angles.

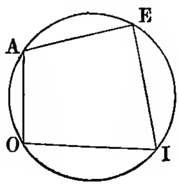
INSCRIBED QUADRILATERAL.

347. Problem.—*Any four points of a circumference may be joined by chords, thus making an inscribed quadrilateral.*

This is a corollary of Article 47.

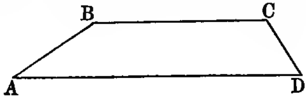
348. Theorem.—*The opposite angles of an inscribed quadrilateral are supplementary.*

For the angle A is measured by half of the arc EIO (222), and the angle I by half of the arc EAO. Therefore, the two together are measured by half of the whole circumference, and their sum is equal to two right angles (207).



TRAPEZOID.

349. If two adjacent angles of a quadrilateral are supplemental, the remaining angles are also supplemental (346). Then, one pair of opposite sides must be parallel (131).

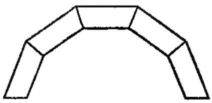


A TRAPEZOID is a quadrilateral which has two sides parallel. The parallel sides are called its *bases*.

350. Corollary.—If the angles adjacent to one base of a trapezoid be equal, those adjacent to the other base must also be equal. For if A and D are equal, their supplements, B and C, must be equal (96).

APPLICATION.

351. The figure described in the last corollary is symmetrical. For it can be divided into equal parts by a line joining the middle points of the bases.



The symmetrical trapezoid is used in architecture, sometimes for ornament, and sometimes as the form of the stones of an arch.

EXERCISES.

352.—1. To construct a quadrilateral when the four sides and one diagonal are given. For example, the side AB, 2 inches; the side BC, 5; CD, 3; DA, 4; and the diagonal AC, 6 inches.

2. To construct a quadrilateral when the four sides and one angle are given.

3. In a quadrilateral, join any point on one side to each end of the side opposite, and with the figure thus constructed demonstrate the theorem, Article 346.

4. The sum of two opposite sides of any quadrilateral which is

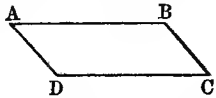
circumscribed about a circle, is equal to the sum of the other two sides.

5. If the two oblique sides of a trapezoid be produced till they meet, then the point of meeting, the point of intersection of the two diagonals of the trapezoid, and the middle points of the two bases are all in one straight line.

PARALLELOGRAMS.

353. A PARALLELOGRAM is a quadrilateral which has its opposite sides parallel.

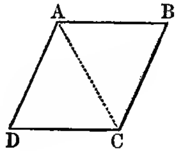
354. Corollary.—Two adjacent angles of a parallelogram are supplementary. The angles A and B, being between the parallels AD and BC, and on one side of the secant AB, are supplementary (126).



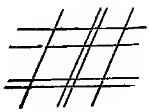
355. Corollary.—The opposite angles of a parallelogram are equal. For both D and B are supplements of the angle C (96).

356. Theorem.—*The opposite sides of a parallelogram are equal.*

For, joining AC by a diagonal, the triangles thus formed have the side AC common; the angles ACB and DAC equal, for they are alternate (125); and ACD and BAC equal, for the same reason. Therefore (285), the triangles are equal, and the side AD is equal to BC, and AB to CD.



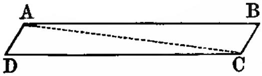
357. Corollary.—When two systems of parallels cross each other, the parts of one system included between two lines of the other are equal.



358. Corollary.—A diagonal divides a parallelogram into two equal triangles. But the diagonal does not divide the figure symmetrically, because the position of the sides of the triangles is reversed.

359. Theorem.—*If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

Join AC. Then, the triangles ABC and CDA are equal. For the side AD is equal to BC, and DC is equal to AB, by hypothesis; and they have the side AC common.

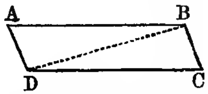


Therefore, the angles DAC and BCA are equal. But these angles are alternate with reference to the lines AD and BC, and the secant AC. Hence, AD and BC are parallel (130), and, for a similar reason, AB and DC are parallel. Therefore, the figure is a parallelogram.

360. Theorem.—*If, in a quadrilateral, two opposite sides are equal and parallel, the figure is a parallelogram.*

If AD and BC are both equal and parallel, then AB is parallel to DC.

For, joining BD, the triangles thus formed are equal, since they have the side BD common, the side AD equal to

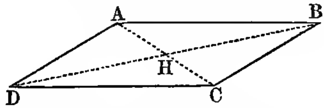


BC, and the angle ADB equal to its alternate DBC (284). Hence, the angle ABD is equal to BDC. But these are alternate with reference to the lines AB and DC, and the secant BD.

Therefore, AB and DC are parallel, and the figure is a parallelogram.

361. Theorem.—*The diagonals of a parallelogram bisect each other.*

The diagonals AC and BD are each divided into equal parts at H, the point of intersection.



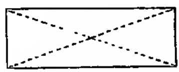
For the triangles ABH and CDH have the sides AB and CD equal (356), the angles ABH and CDH equal (125), and the angles BAH and DCH equal. Therefore, the triangles are equal (285), and AH is equal to CH, and BH to DH.

362. Theorem.—*If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

To be demonstrated by the student.

RECTANGLE

363. If one angle of a parallelogram is right, the others must be right also (354).

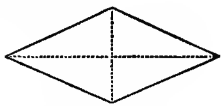


A RECTANGLE is a right angled parallelogram. The rectangle has all the properties of other parallelograms, and the following peculiar to itself, which the student may demonstrate.

364. Theorem.—*The diagonals of a rectangle are equal.*

RHOMBUS.

365. When two adjacent sides of a parallelogram are equal, all its sides must be equal (356).



A RHOMBUS, or, as sometimes called, a LOZENGE, is a parallelogram having all its sides equal.

The rhombus has the following peculiarities, which may be demonstrated by the student.

366. Theorem.—*The diagonals of a rhombus are perpendicular to each other.*

367. Theorem.—*The diagonals of a rhombus bisect its angles.*

SQUARE.

368. A SQUARE is a quadrilateral having its sides equal, and its angles right angles. The square may be shown to have all the properties of the parallelogram (359), of the rectangle, and of the rhombus.

369. Corollary.—The rectangle and the square are the only parallelograms which can be inscribed in a circle (348).

EQUALITY.

370. Theorem.—*Two parallelograms are equal when two adjacent sides and the included angle in the one, are respectively equal to those parts in the other.*

For the remaining sides must be equal (356), and this becomes a case of Article 345.

371. Corollary.—Two rectangles are equal when two adjacent sides of the one, are respectively equal to those parts of the other.

372. Corollary.—Two squares are equal when a side of the one is equal to a side of the other.

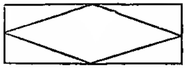
APPLICATIONS.

373. The rectangle is the most frequently used of all quadrilaterals. The walls and floors of our apartments, doors and windows, books, paper, and many other articles, have this form.

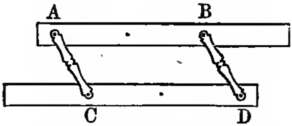
Carpenters make an ingenious use of a geometrical principle in order to make their door and window-frames exactly rectangular. Having made the frame, with its sides equal and its ends equal, they measure the two diagonals, and make the frame take such a shape that these also will be equal.

In this operation, what principle is applied?

374. A rhombus inscribed in a rectangle is the basis of many ornaments used in architecture and other work.



375. An instrument called *parallel rulers*, used in drawing parallel lines, consists of two rulers, connected by cross pieces with pins in their ends. The rulers may turn upon the pins, varying their distance. The distances between the pins along the rulers, that is, AB and CD, must be equal; also, along the cross pieces, that is, AC and BD. Then the rulers will always be parallel to each other. If one ruler be held fast while the other is moved, lines drawn along the edge of the other ruler, at different positions, will be parallel to each other.



What geometrical principles are involved in the use of this instrument?

EXERCISES.

376.—1. State the converse of each theorem that has been given in this chapter, and determine whether each of these converse propositions is true.

2. To construct a parallelogram when two adjacent sides and an angle are given.

3. What parts need be given for the construction of a rectangle?

4. What must be given for the construction of a square?

5. If the four middle points of the sides of any quadrilateral be joined by straight lines, those lines form a parallelogram.

6. If four points be taken, one in each side of a square, at equal distances from the four vertices, the figure formed by joining these successive points is a square.

7. Two parallelograms are similar when they have an angle in the one equal to an angle in the other, and these equal angles included between proportional sides.

MEASURE OF AREA.

377. The standard figure for the measure of surfaces is a square. That is, the unit of area is a square, the side of which is the unit of length, whatever be the extent of the latter.

Other figures might be, and sometimes are, used for this purpose; but the square has been almost universally adopted, because

1. Its form is regular and simple;

2. The two dimensions of the square, its length and breadth, are the same; and,

3. A plane surface can be entirely covered with equal squares.

The truth of the first two reasons is already known to the student: that of the last will appear in the following theorem.

378. Any side of a polygon may be taken as the *base.*

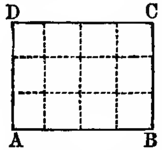
The ALTITUDE of a parallelogram is the distance between the base and the opposite side. Hence, the altitude of a parallelogram may be taken in either of two ways.

AREA OF RECTANGLES.

379. Theorem.—*The area of a rectangle is measured by the product of its base by its altitude.*

That is, if we multiply the number of units of length contained in the base, by the number of those units contained in the altitude, the product is the number of units of area contained in the surface.

Suppose that the base AB and the altitude AD are multiples of the same unit of length, for example, four and three. Divide AB into four equal parts, and through all the points of division extend lines parallel to AD. Divide AD into three equal parts, and through the points of division extend lines parallel to AB.



All the intercepted parts of these two sets of parallels must be equal (357); and all the angles, right angles (124). Thus, the whole rectangle is divided into equal squares (372). The number of these squares is equal to the number in one row multiplied by the number of rows; that is, the number of units of length in the base multiplied by the number in the altitude. In the example taken, this is three times four, or twelve. The result would be the same, whatever the number of divisions in the base and altitude.

If the base and altitude have no common measure, then we may assume the unit of length as small as we please. By taking for the unit a less and less part of the altitude, the base will be made the limit of the lines commensurable with the altitude. Thus, the demonstration is made general.

380. Corollary.—The area of a square is expressed by the second power of the length of its side. Anciently the principles of arithmetic were taught and illustrated by geometry, and we still find the word *square* in common use for the second power of a number.

381. By the method of infinites (203), the latter part of the above demonstration would consist in supposing the base and altitude of the rectangle divided into infinitely small and equal parts; and then proceeding to form infinitesimal squares, as in the former part of the demonstration.

If a straight line move in a direction perpendicular to itself, it describes a rectangle, one of whose dimensions is the given line, and the other is the distance which it has moved. Thus, it appears that the two dimensions which every surface has (33), are combined in the simplest manner in the rectangle.

A rectangle is said to be *contained* by its base and altitude. Thus, also, the area of any figure is called its *superficial contents.*

APPLICATION.

382. All enlightened nations attach great importance to exact and uniform standard measures. In this country the standard of length is a yard measure, carefully preserved by the National Government, at Washington City. By it all the yard measures are regulated.

The standards generally used for the measure of surface, are the square described upon a yard, a foot, a mile, or some other certain length; but the acre, one of the most common measures of surface, is an exception. The number of feet, yards, or rods in one side of a square acre, can only be expressed by the aid of a radical sign.

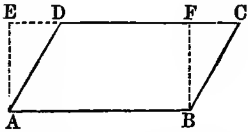
The public lands belonging to the United States are divided into square townships, each containing thirty-six square miles, called *sections.*

AREA OF PARALLELOGRAMS.

383. *The area of a parallelogram is measured by the product of its base by its altitude.*

At the ends of the base AB erect perpendiculars, and produce them till they meet the opposite side, in the points E and F.

Now the right angled triangles AED and BFC are equal, having the side BF equal to AE, since they are perpendiculars between parallels (133); and the side BC equal to AD, by hypothesis (288). If each of these equal triangles be subtracted from the entire figure, ABCE, the remainders ABFE and ABCD must be equivalent. But ABFE is a rectangle having the same base and altitude as the parallelogram ABCD. Hence, the area of the parallelogram is measured by the same product as that which measures the area of the rectangle.



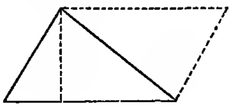
384. Corollary.—Any two parallelograms have their areas in the same ratio as the products of their bases by their altitudes. Parallelograms of equal altitudes have the same ratio as their bases, and parallelograms of equal bases have the same ratio as their altitudes.

385. Corollary.—Two parallelograms are equivalent when they have equal bases and altitudes; or, when the two dimensions of the one are the extremes, and the two dimensions of the other are the means, of a proportion.

AREA OF TRIANGLES.

386. Theorem.—*The area of a triangle is measured by half the product of its base by its altitude.*

For any triangle is one-half of a parallelogram having the same base and altitude (358).



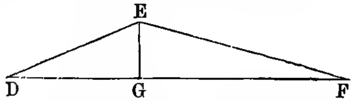
387. Corollary.—The areas of triangles are in the ratio of the products of their bases by their altitudes.

388. Corollary.—Two triangles are equivalent when they have equal bases and altitudes.

389. Corollary.—If a parallelogram and a triangle have equal bases and altitudes, the area of the parallelogram is double that of the triangle.

390. Theorem.—*If from half the sum of the three sides of a triangle each side be subtracted, and if these remainders and the half sum be multiplied together, then the square root of the product will be the area of the triangle.*

Let DEF be any triangle, DF being the base and EG the altitude. Let the extent of the several lines be represented by letters; that is, let



DF $=a$, EF $=b$, DE $=c$, EG $=h$, GF $=m$, DG $=n$, and DE + EF + FD $=s$.

Then (328), $m+n : b+c :: b-c : m-n$.

Therefore, $$m-n=\frac{b^2-c^2}{m+n}.$$

By hypothesis, $$m+n=a.$$

Adding, $$2m=a+\frac{b^2-c^2}{a}.$$

Then, $$m=\frac{a^2+b^2-c^2}{2a}.$$

Again (327), $$m^2+h^2=b^2.$$

Substituting for m^2 its value, and transposing,

$$h^2=b^2-\left(\frac{a^2+b^2-c^2}{2a}\right)^2.$$

Therefore, $$h=\sqrt{b^2-\left(\frac{a^2+b^2-c^2}{2a}\right)^2}.$$

But the area of the triangle is half the product of the base a by the altitude h. Hence,

$$area\ \text{DEF} = \frac{ah}{2} = \frac{a}{2}\sqrt{b^2 - \left(\frac{a^2+b^2-c^2}{2a}\right)^2}.$$

In this expression, we have the area of the triangle in terms of the three sides. For greater facility of calculation it is reduced to the following:

$$area = \tfrac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}.$$

The exact equality of these two expressions is shown by performing as far as is possible the operations indicated in each.

But, by hypothesis, $(a+b+c) = s = 2\left(\frac{s}{2}\right)$.

Therefore, $(a+b-c) = 2\left(\frac{s}{2} - c\right)$,

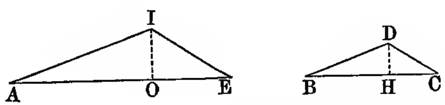
$(a-b+c) = 2\left(\frac{s}{2} - b\right)$,

and, $(-a+b+c) = 2\left(\frac{s}{2} - a\right)$.

Substituting these in the equation of the area, it becomes,

$$area = \sqrt{\left(\frac{s}{2}\right)\left(\frac{s}{2} - a\right)\left(\frac{s}{2} - b\right)\left(\frac{s}{2} - c\right)}.$$

391. Theorem.—*The areas of similar triangles are to each other as the squares of their homologous lines.*



Let AEI and BCD be similar triangles, and IO and DH homologous altitudes.

Then (310), $IO : DH :: AE : BC$.

Multiply by $AE : BC :: AE : BC$.

Then, $AE \times IO : BC \times DH :: \overline{AE}^2 : \overline{BC}^2$.

But (387),

$$AE \times IO : BC \times DH :: \text{area } AEI : \text{area } BCD.$$

Therefore (21),

$$\text{area } AEI : \text{area } BCD :: \overline{AE}^2 : \overline{BC}^2.$$

In a similar manner, prove that the areas have the same ratio as the squares of the altitudes IO and DH, or as the squares of any homologous lines.

AREA OF TRAPEZOIDS.

392. Theorem.—*The area of a trapezoid is equal to half the product of its altitude by the sum of its parallel sides.*

The trapezoid may be divided by a diagonal into two triangles, having for their bases the parallel sides.

The altitude of each of these triangles is equal to that of the trapezoid (264). The area of each triangle being half the product of the common altitude by its base, the area of their sum, or of the whole trapezoid, is half the product of the altitude by the sum of the bases.

EXERCISES.

393.—1. Measure the length and breadth, and find the area of the blackboard; of the floor.

2. To divide a given triangle into any number of equivalent triangles.

3. To divide a given parallelogram into any number of equivalent parallelograms.

4. To divide a given trapezoid into any number of equivalent trapezoids.

5. The area of a triangle is equal to half the product of the perimeter by the radius of the inscribed circle.

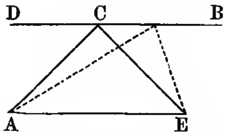
6. What is the radius of the circle inscribed in the triangle whose sides are 8, 10, and 12?

EQUIVALENT SURFACES.

394. Isoperimetrical figures are those whose perimeters have the same extent.

395. Theorem.—*Of all equivalent triangles of a given base, the one having the least perimeter is isosceles.*

The equivalent triangles having the same base, AE, have also the same altitude (388). Hence, their vertices are in the same line parallel to the base, that is, in DB.



Now, the shortest line that can be made from A to E through some point of DB, will constitute the other two sides of the triangle of least perimeter. This shortest line is the one making equal angles with DB, as ACE, that is, making ACD and ECB equal (115). The angle ACD is equal to its alternate A, and the angle ECB to its alternate E. Therefore, the angles at the base are equal, and the triangle is isosceles.

396. Corollary.—Of all isoperimetrical triangles of a given base, the one having the greatest area is isosceles.

397. To draw a square equivalent to a given figure, is called the *squaring*, or *quadrature* of the figure. How this can be done for any rectilinear figure, is shown in the following.

PROBLEMS IN DRAWING.

398. Problem.—*To draw a rectangle with a given base, equivalent to a given parallelogram.*

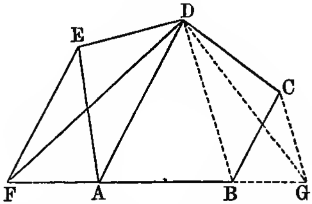
With the given base as a first term, and the base and altitude of the given figure as the second and third terms, find a fourth proportional (319). This is the required altitude (385).

399. Problem.—*To draw a square equivalent to a given parallelogram.*

Find a mean proportional between the base and altitude of the given figure (330). This is the side of the square (385).

400. Problem.—*To draw a triangle equivalent to a given polygon.*

Let ABCDE be the given polygon. Join DA. Produce BA, and through E draw EF parallel to DA. Join DF.



Now, the triangles DAF and DAE are equivalent, for they have the same base DA, and equal altitudes, since their vertices are in the line EF parallel to the base (264). To each of these equals, add the figure ABCD, and we have the quadrilateral FBCD equivalent to the polygon ABCDE. In this manner, the number of sides may be diminished till a triangle is formed equivalent to the given polygon. In this diagram it is the triangle FDG.

401. Problem.—*To draw a square equivalent to a given triangle.*

Find a mean proportional between the altitude and half the base of the triangle. This will be the side of the required square.

EQUIVALENT SQUARES.

402. Having shown (379) how an area is expressed by the product of two lengths, it follows that an equation will represent equivalent surfaces, if each of its terms is composed of two factors which represent lengths.

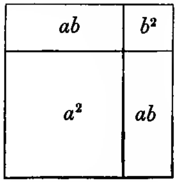
For example, let a and b represent the lengths of two straight lines. Now we know, from algebra, that whatever be the value of a and b,

$$(a+b)^2 = a^2 + 2ab + b^2.$$

This formula, therefore, includes the following geometrical

403. Theorem.—*The square described upon the sum of two lines is equivalent to the sum of the squares described on the two lines, increased by twice the rectangle contained by these two lines.*

Since the truths of algebra are universal in their application, this theorem is demonstrated by the truth of the above equation.



Such a proof is called *algebraic.* It is also called *analytical,* but with doubtful propriety.

Let the student demonstrate the theorem geometrically, by the aid of this diagram.

404. Theorem.—*The square described on the difference of two straight lines is equivalent to the sum of the squares described on the two lines, diminished by twice the rectangle contained by those lines.*

This is a consequence of the truth of the equation,

$$(a-b)^2 = a^2 - 2ab + b^2.$$

405. Theorem.—*The rectangle contained by the sum and the difference of two straight lines is equivalent to the difference of the squares of those lines.*

This, again, is proved by the principle expressed in the equation,

$$(a+b)\ (a-b)=a^2-b^2.$$

406. These two theorems may also be demonstrated by purely geometrical reasoning.

The algebraic method is sometimes called the modern, while the other is called the ancient geometry. The algebraic method was invented by Descartes, in the seventeenth century, while the other is twenty centuries older.

THE PYTHAGOREAN THEOREM.

407. Since numerical equations represent geometrical truths, the following theorem might be inferred from Article 327.

This is called the *Pythagorean* Theorem, because it was discovered by Pythagoras. It is also known as the *Forty-seventh* Proposition, that being its number in the First Book of Euclid's Elements.

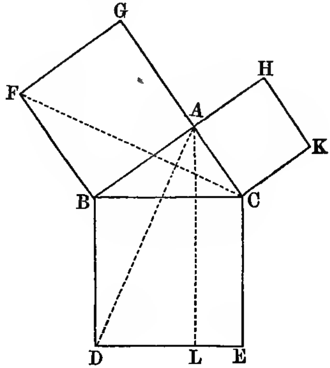
It has been demonstrated in a great variety of ways. One is by dividing the three squares into parts, so that the several parts of the large square are respectively equal to the several parts of the two others.

The fame of this theorem makes it proper to give here the demonstration from Euclid.

408. Theorem.—*The square described on the hypotenuse of a right angled triangle is equivalent to the sum of the squares described on the two legs.*

Let ABC be a right angled triangle, having the right angle BAC. The square described on the side BC is equivalent to the sum of the two squares described on BA and AC. Through A make AL parallel to BD, and join AD and FC.

Then, because each of the angles BAC and BAG is a right angle, the line GAC is one straight line (100). For the same reason, BAH is one straight line.



The angles FBC and DBA are equal, since each is the sum of a right angle and the angle ABC. The two triangles FBC and DBA are equal, for the side FB in the one is equal to BA in the other, and the side BC in the one is equal to BD in the other, and the included angles are equal, as just proved.

Now, the area of the parallelogram BL is double that of the triangle DBA, because they have the same base BD, and the same altitude DL (389). And the area of the square BG is double that of the triangle FBC, because these also have the same base BF, and the same altitude FG. But doubles of equals are equal (7). Therefore, the parallelogram BL and the square BG are equivalent.

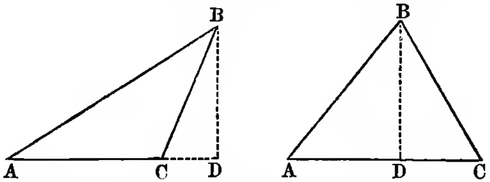
In the same manner, by joining AE and BK, it is demonstrated that the parallelogram CL and the square CH are equivalent. Therefore, the whole square BE, described on the hypotenuse, is equivalent to the two squares BG and CH, described on the legs of the right angled triangle.

409. Corollary.—The square described on one leg is equivalent to the difference of the squares on the hypotenuse and the other leg.

410. If from the extremities of one line perpendiculars be let fall upon another, then the part of the second line between the perpendiculars is called the *projection* of the first line on the second. If one end of the first line is in the second, then only one perpendicular is necessary.

411. Theorem.—*The square described on the side opposite to an acute angle of a triangle, is equivalent to the sum of the squares described on the other two sides, diminished by twice the rectangle contained by one of these sides and the projection of the other on that side.*

Let A be the acute angle, and from B let a perpendicular fall upon AC, produced if necessary. Then,



AD is the projection of AB upon AC. And it is to be proved that the square on BC is equivalent to the sum of the squares on AB and on AC, diminished by twice the rectangle contained by AC and AD.

For (409), $\overline{BD}^2 = \overline{AB}^2 - \overline{AD}^2$;

and (404), $\overline{CD}^2 = \overline{AC}^2 + \overline{AD}^2 - 2AC \times AD.$

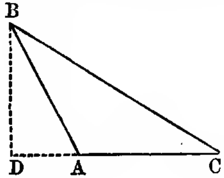
By addition, $\overline{BD}^2 + \overline{CD}^2 = \overline{AB}^2 + \overline{AC}^2 - 2AC \times AD.$

But the square on BC is equivalent to $\overline{BD}^2 + \overline{CD}^2$ (408). Therefore, it is also equivalent to

$$\overline{AB}^2 + \overline{AC}^2 - 2AC \times AD.$$

412. Theorem.—*The square described on the side opposite an obtuse angle of a triangle, is equivalent to the sum of the squares described on the other two sides, increased by twice the rectangle of one of those sides and the projection of the other on that side.*

In the triangle ABC, the square on BC which is opposite the obtuse angle at A, is equivalent to the sum of the squares on AB and on AC, and twice the rectangle contained by CA and AD.



For, $$\overline{BD}^2 = \overline{AB}^2 - \overline{AD}^2;$$

and (403), $$\overline{CD}^2 = \overline{AC}^2 + \overline{AD}^2 + 2AC \times AD.$$

By addition, $$\overline{BD}^2 + \overline{CD}^2 = \overline{AB}^2 + \overline{AC}^2 + 2AC \times AD.$$

But, $$\overline{BC}^2 = \overline{BD}^2 + \overline{CD}^2.$$

Therefore, $\overline{BC}^2$ is equivalent to

$$\overline{AB}^2 + \overline{AC}^2 + 2AC \times AD.$$

413. Corollary.—If the square described on one side of a triangle is equivalent to the sum of the squares described on the other two sides, then the opposite angle is a right angle. For the last two theorems show that it can be neither acute nor obtuse.

EXERCISES.

414.—1. When a quadrilateral has its opposite angles supplementary, a circle can be circumscribed about it.

2. From a given isosceles triangle, to cut off a trapezoid which

shall have the same base as the triangle, and the remaining three sides equal to each other.

3. The lines which bisect the angles of a parallelogram, form a rectangle whose diagonals are parallel to the sides of the parallelogram.

4. In any parallelogram, the distance of one vertex from a straight line passing through the opposite vertex, is equal to the sum or difference of the distances of the line from the other two vertices, according as the line is without or within the parallelogram.

5. When one diagonal of a quadrilateral divides the figure into equal triangles, is the figure necessarily a parallelogram?

6. Demonstrate the theorem, Article 329, by Articles 113 and 387.

7. What is the area of a lot, which has the shape of a right angled triangle, the longest side being 100 yards, and one of the other sides 36 yards.

8. Can every triangle be divided into two equal parts? Into three? Into nine?

9. Two parallelograms having the same base and altitude are equivalent.

To be demonstrated without using Articles 379 or 383.

10. A triangle is divided into two equivalent parts, by a line from the vertex to the middle of the base.

To be demonstrated without the aid of the principles of this chapter.

11. To divide a triangle into two equivalent parts, by a line drawn from a given point in one of the sides.

12. Of all equivalent parallelograms having equal bases, what one has the minimum perimeter?

13. Find the locus of the points such that the sum of the squares of the distances of each from two given points, shall be equivalent to the square of the line joining the given points.

CHAPTER VII.

POLYGONS.

415. Hitherto the student's attention has been given to polygons of three and of four sides only. He has seen how the theories of similarity and of linear ratio have grown out of the consideration of triangles; and how the study of quadrilaterals gives us the principles for the measure of surfaces, and the theory of equivalent figures.

In the present chapter, some principles of polygons of any number of sides will be established.

A *Pentagon* is a polygon of five sides; a *Hexagon* has six sides; an *Octagon*, eight; a *Decagon*, ten; a *Dodecagon*, twelve; and a *Pentedecagon*, fifteen.

The following propositions on diagonals, and on the sum of the angles, are more general statements of those in Articles 340 to 346.

DIAGONALS.

416. Theorem.—*The number of diagonals from any vertex of a polygon, is three less than the number of sides.*

For, from each vertex a diagonal may extend to every other vertex except itself, and the one adjacent on each side. Thus, the number is three less than the number of vertices, or of sides.

417. Corollary.—The diagonals from one vertex divide a polygon into as many triangles as the polygon has sides, less two.

Polygons may be divided into this number, or into a greater number of triangles, in various ways; but a polygon can not be divided into a less number of triangles than here stated.

418. Corollary.—The whole number of diagonals possible in a polygon of n sides, is $\frac{1}{2}\, n\, (n - 3)$. For, if we count the diagonals at all the n vertices, we have $n\, (n - 3)$, but this is counting each diagonal at both ends. This last product must therefore be divided by two.

EQUAL POLYGONS.

419. Theorem.—*Two polygons are equal when they are composed of the same number of triangles respectively equal and similarly arranged.*

This is an immediate consequence of the definition of equality (40).

420. Corollary.—Conversely, two equal polygons may be divided into the same number of triangles respectively equal and similarly arranged.

421. Theorem.—*Two polygons are equal when all the sides and all the diagonals from one vertex of the one, are respectively equal to the same lines in the other, and are similarly arranged.*

For each triangle in the one would have its three sides equal to the similarly situated triangle in the other, and would be equal to it (282). Therefore, the polygons would be equal (419).

422. Theorem.—*Two polygons are equal when all the sides and the angles of the one are respectively equal to the same parts of the other, and are similarly arranged.*

For each triangle in the one is equal to its homologous triangle in the other, since they have two sides and the included angle equal.

It is enough for the hypothesis of this theorem, that all the angles except three be among the equal parts.

SUM OF THE ANGLES.

423. Theorem.—*The sum of all the angles of a polygon is equal to twice as many right angles as the polygon has sides, less two.*

For the polygon may be divided into as many triangles as it has sides, less two (417); and the angles of these triangles coincide altogether with those of the polygon.

The sum of the angles of each triangle is two right angles. Therefore, the sum of the angles of the polygon is equal to twice as many right angles as it has sides, less two.

The remark in Article 346 applies as well to this theorem.

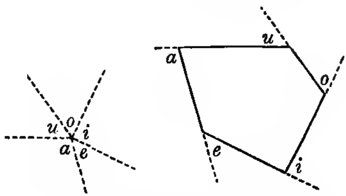
424. Let R represent a right angle; then the sum of the angles of a polygon of n sides is $2\,(n-2)$ R; or, it may be written thus, $(2n-4)$ R.

The student should illustrate each of the last five theorems with one or more diagrams.

425. Theorem.—*If each side of a convex polygon be produced, the sum of all the exterior angles is equal to four right angles.*

Let the sides be produced all in one way; that is, all to the right or all to the left. Then, from any point in the plane, extend lines parallel to the sides thus produced, and in the same directions.

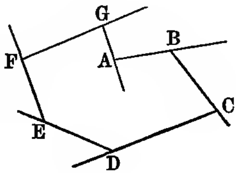
The angles thus formed are equal in number to the exterior angles of the polygon, and are respectively equal to them (138). But the sum of those formed about the point is equal to four right angles (92).



Therefore, the sum of the exterior angles of the polygon is equal to four right angles.

426. This theorem will also be true of concave polygons, if the angle formed by producing one side of the reëntrant angle is considered as a negative quantity.

Thus, the remainder, after subtracting the angle formed at A by producing GA, from the sum of the angles formed at B, C, D, E, F, and G, is four right angles. This may be demonstrated by the aid of the previous theorem (423).



EXERCISES.

427.—1. What is the number of diagonals that can be in a pentagon? In a decagon?

2. What is the sum of the angles of a hexagon? Of a dodecagon?

3. What is the greatest number of acute angles which a convex polygon can have?

4. Join any point within a given polygon with every vertex of the polygon, and with the figure thus formed, demonstrate the theorem, Article 423.

5. Demonstrate the theorem, Article 425, by means of Article 23, and without using Article 92.

PROBLEMS IN DRAWING.

428. Problem.—*To draw a polygon equal to a given polygon.*

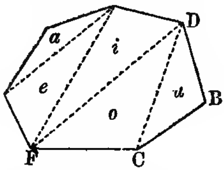
By diagonals divide the given polygon into triangles. The problem then consists in drawing triangles equal to given triangles.

429. Problem.—*To draw a polygon when all its sides and all the diagonals from one vertex, are given in their proper order.*

This consists in drawing triangles with sides equal to three given lines (295).

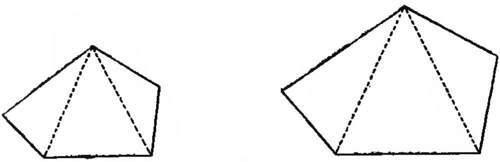
430. Problem.—*To draw a polygon when the sides and angles are given in their order.*

It is enough for this problem if all the angles except three be given. For, suppose first that the angles not given are consecutive, as at D, B, and C. Then, draw the triangles *a*, *e*, *i*, and *o* (297). Then, having DC, complete the polygon by drawing the triangle DBC from its three known sides (295). Suppose the angles not given were D, C, and F. Then, draw the triangles *a*, *e*, and *i*, and separately, the triangle *u*. Then, having the three sides of the triangle *o*, it may be drawn, and the polygon completed.



SIMILAR POLYGONS.

431. Theorem.—*Similar polygons are composed of the same number of triangles, respectively similar and similarly arranged.*



Since the figures are similar, every angle in one has its corresponding equal angle in the other (303). If, then, diagonals be made to divide one of the polygons into triangles, every angle thus formed may have its corresponding equal angle in the other. Therefore, the triangles of one polygon are respectively similar to those of the other, and are similarly arranged.

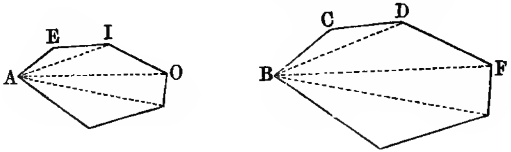
432. Theorem.—*If two polygons are composed of the same number of triangles which are respectively similar and are similarly arranged, the polygons are similar.*

By the hypothesis, all the angles formed by the given lines in one polygon have their corresponding equal angles in the other. It remains to be proved that angles formed by any other lines in the one have their corresponding equal angles in the other polygon.

This may be shown by reasoning, in the same manner as in the case of triangles (304). Let the student make the diagrams and complete the demonstration.

433. Theorem.—*Two polygons are similar when the angles formed by the sides are respectively equal, and there is the same ratio between each side of the one and its homologous side of the other.*

Let all the diagonals possible extend from a vertex A



of one polygon, and the same from the homologous vertex B of the other polygon.

Now the triangles AEI and BCD are similar, because they have two sides proportional, and the included angles equal (317).

Therefore, EI : CD : : AI : BD.

But, by hypothesis, EI : CD : : IO : DF.

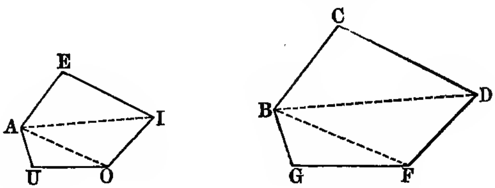
Then (21), AI : BD : : IO : DF.

Also, if we subtract the equal angles EIA and CDB from the equal angles EIO and CDF, the remainders AIO and BDF are equal. Hence, the triangles AIO and BDF are similar. In the same manner, prove that each of the triangles of the first polygon is similar to its corresponding triangle in the other. Therefore, the figures are similar (432).

As in the case of equal polygons (422 and 430), it is only necessary to the hypothesis of this proposition, that all the angles except three in one polygon be equal to the homologous angles in the other.

434. Theorem.—*In similar polygons the ratio of two homologous lines is the same as of any other two homologous lines.*

For, since the polygons are similar, the triangles which



compose them are also similar, and (309),

AE : BC : : EI : CD : : AI : BD : : IO : DF, etc.

This common ratio is the linear ratio of the two figures.

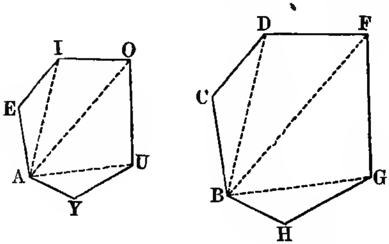
Let the student show that the perpendicular let fall from E upon OU, and the homologous line in the other polygon, have the linear ratio of the two figures.

435. Theorem.—*The perimeters of similar polygons are to each other as any two homologous lines.*

The student may demonstrate this theorem in the same manner as the corresponding propositions in triangles (312).

436. Theorem.—*The area of any polygon is to the area of a similar polygon, as the square on any line of the first is to the square on the homologous line of the second.*

Let the polygons BCD, etc., and AEI, etc., be divided into triangles by homologous diagonals. The triangles thus formed in the one are similar to those formed in the other (431).



Therefore (391),

area BCD : area AEI : : $\overline{BD}^2$: $\overline{AI}^2$: : area BDF : area AIO : : $\overline{BF}^2$: $\overline{AO}^2$: : area BFG : area AOU : : $\overline{BG}^2$: $\overline{AU}^2$: : area BGH : area AUY.

Selecting from these equal ratios the triangles, area BCD : area AEI : : area BDF : area AIO : : area BFG : area AOU : : area BGH : area AUY.

Therefore (23), area BCDFGHB : area AEIOUYA : : area BCD : area AEI; or, as $\overline{BC}^2$: $\overline{AE}^2$; or, as the areas of any other homologous parts; or, as the squares of any other homologous lines.

437. Corollary.—The superficial ratio of two similar polygons is always the second power of their linear ratio.

EXERCISES.

438.—1. Compose two polygons of the same number of triangles respectively similar, but not similarly arranged.

2. To draw a triangle similar to a given triangle, but with double the area.

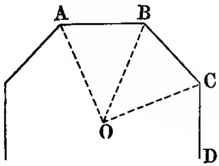
3. What is the relation between the areas of the equilateral triangles described on the three sides of a right angled triangle?

REGULAR POLYGONS.

439. A Regular Polygon is one which has all its sides equal, and all its angles equal. The square and the equilateral triangle are regular polygons.

440. Theorem.—*Within a regular polygon there is a point equally distant from the vertices of all the angles.*

Let ABCD, etc., be a regular polygon, and let lines bisecting the angles A and B extend till they meet at O. These lines will meet, for the interior angles which they make with AB are both acute (137).



In the triangle ABO, the angles at A and B are equal, being halves of the equal angles of the polygon. Therefore, the opposite sides AO and BO are equal (275).

Join OC. Now, the triangles ABO and BCO are equal, for they have the side AO of the first equal to BO of the second, the side AB equal to BC, because the polygon is regular, and the included angles OAB and OBC equal, since they are halves of angles of the polygon. Hence, BO is equal to OC.

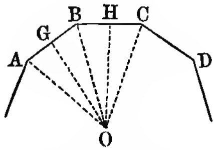
Then, the angle OCB is equal to OBC (268), and OC bisects the angle BCD, which is equal to ABC. In the same manner, it is proved that OC is equal to OD, and so on. Therefore, the point O is equally distant from all the vertices.

CIRCUMSCRIBED AND INSCRIBED.

441. Corollary.—Every regular polygon may have a circle circumscribed about it. For, with O as a center and OA as a radius, a circumference may be described passing through all the vertices of the polygon (153).

442. Theorem.—*The point which is equally distant from the vertices is also equally distant from the sides of a regular polygon.*

The triangles OAB, OBC, etc., are all isosceles. If perpendiculars be let fall from O upon the several sides AB, BC, etc., these sides will be bisected (271). Then, the perpendiculars will be equal, for they will be sides of equal triangles. But they measure the distances from O to the several sides of the polygon. Therefore, the point O is equally distant from all the sides of the polygon.



443. Corollary.—Every regular polygon may have a circle inscribed in it. For with O as a center and OG as a radius, a circumference may be described passing through the feet of all these perpendiculars, and tangent to all the sides of the polygon (178), and therefore inscribed in it (253).

444. Corollary.—A regular polygon is a symmetrical figure.

445. The center of the circumscribed or inscribed circle is also called the *center of a regular polygon.* The radius of the circumscribed circle is also called the *radius of a regular polygon.*

The APOTHEM of a regular polygon is the radius of the inscribed circle.

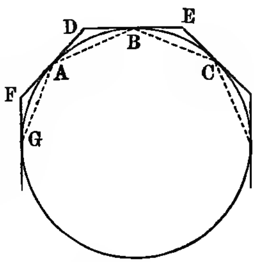
446. Theorem.—*If the circumference of a circle be divided into equal arcs, the chords of those equal arcs will be the sides of a regular polygon.*

For the sides are all equal, being the chords of equal arcs (185); and the angles are all equal, being inscribed in equal arcs (224).

447. Corollary.—An angle formed at the center of a regular polygon by lines from adjacent vertices, is an aliquot part of four right angles, being the quotient of four right angles divided by the number of the sides of the polygon.

448. Theorem.—*If a circumference be divided into equal arcs, and lines tangent at the several points of division be produced until they meet, these tangents are the sides of a regular polygon.*

Let A, B, C, etc., be points of division, and F, D, and E points where the tangents meet.



Join GA, AB, and BC.

Now, the triangles GAF, ABD, and BCE have the sides GA, AB, and BC equal, as they are chords of equal arcs; and the angles at G, A, B, and C equal, for each is formed by a tangent and chord which intercept equal arcs (226). Therefore, these triangles are all isosceles (275), and all equal (285); and the angles F, D, and E are equal. Also, FD and DE, being doubles of equals, are equal. In the same manner, it is proved that all the angles of the polygon FDE, etc., are equal, and that all its sides are equal. Therefore, it is a regular polygon.

REGULAR POLYGONS SIMILAR.

449. Theorem.—*Regular polygons of the same number of sides are similar.*

Since the polygons have the same number of sides, the sum of all the angles of the one is equal to the sum of all the angles of the other (423). But all the angles of a regular polygon are equal (439). Dividing the equal sums by the number of angles (7), it follows that an angle of the one polygon is equal to an angle of the other.

Again: all the sides of a regular polygon are equal. Hence, there is the same ratio between a side of the first and a side of the second, as between any other side of the first and a corresponding side of the second. Therefore, the polygons are similar (433).

450. Corollary.—The areas of two regular polygons of the same number of sides are to each other as the squares of their homologous lines (436).

451. Corollary.—The ratio of the radius to the side of a regular polygon of a given number of sides, is a constant quantity. For a radius of one is to a radius of any other, as a side of the one is to a side of the other (434). Then, by alternation (19), the radius is to the side of one regular polygon, as the radius is to the side of any other regular polygon of the same number of sides.

452. Corollary.—The same is true of the apothem and side, or of the apothem and radius.

PROBLEMS IN DRAWING.

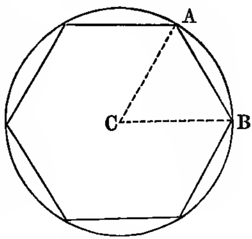
453. Problem.—*To inscribe a square in a given circle.*

Draw two diameters perpendicular to each other. Join their extremities by chords. These chords form an inscribed square.

For the angles at the center are equal by construction (90). Therefore, their intercepted arcs are equal (197), and the chords of those arcs are the sides of a regular polygon (446).

454. Problem.—*To inscribe a regular hexagon in a circle.*

Suppose the problem solved and the figure completed. Join two adjacent angles with the center, making the triangle ABC.



Now, the angle C, being measured by one-sixth of the circumference, is equal to one-sixth of four right angles, or one-third of two right angles. Hence; the sum of the two angles, CAB and CBA, is two-thirds of two right angles (256). But CA and CB are equal, being radii; therefore, the angles CAB and CBA are equal (268), and each of them must be one-third of two right angles. Then, the triangle ABC, being equiangular, is equilateral (276). Therefore, the side of an inscribed regular hexagon is equal to the radius of the circle.

The solution of the problem is now evident—apply the radius to the circumference six times as a chord.

455. Corollary.—Joining the alternate vertices makes an inscribed equilateral triangle.

456. Problem.—*To inscribe a regular decagon in a given circle.*

Divide the radius CA in extreme and mean ratio, at the point B. (334) BC is equal to the side of a regular inscribed decagon. That is, if we apply BC as a chord, its arc will be one-tenth of the whole circumference.

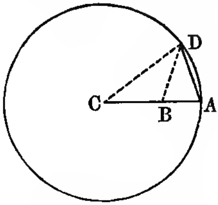
Take AD, making the chord AD equal to BC. Then join DC and DB.

Then, by construction, CA : CB : : CB BA.

Substituting for CB its equal DA,

CA : DA :: DA : BA.

Then the triangles CDA and BDA are similar, for they have those sides proportional which include the common angle A (317). But the triangle CDA being isosceles, the triangle BDA is the same. Hence, DB is equal to DA, and also to BC. Therefore, the angle C is equal to the angle BDC (268). But it is also equal to BDA. It follows that the angle CDA is twice the angle C. The angle at A being equal to CDA, the angle C must be one-fifth of the sum of these three angles; that is, one-fifth of two right angles (255), or one-tenth of four right angles. Therefore, the arc AD is one-tenth of the circumference (207); and the chord AD is equal to the side of an inscribed regular decagon.



457.—**Corollary.**—By joining the alternate vertices of a decagon, we may inscribe a regular pentagon.

458. **Corollary.**—A regular pentedecagon, or polygon of fifteen sides, may be inscribed, by subtracting the arc subtended by the side of a regular decagon from the arc subtended by the side of a regular hexagon. The remainder is one-fifteenth of the circumference, for $\frac{1}{6}-\frac{1}{10}=\frac{1}{15}$.

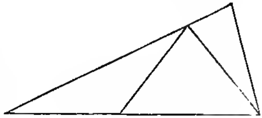
459. Problem.—*Given a regular polygon inscribed in a circle, to inscribe a regular polygon of double the number of sides.*

Divide each arc subtended by a given side into two equal parts (194). Join the successive points into which the circumference is divided. The figure thus formed is the required polygon.

460. We have now learned how to inscribe regular polygons of 3, 4, 5, and 15 sides, and of any number that may arise from doubling either of these four.

The problem, to inscribe a regular polygon in a circle by means of straight lines and arcs of circles, can be solved in only a limited number of cases. It is evident that the solution depends upon the division of the circumference into any number of equal parts; and this depends upon the division of the sum of four right angles into aliquot parts.

461. Notice that the regular decagon was drawn by the aid of two isosceles triangles composing a third, one of the two being similar to the whole. Now, if we could combine three isosceles triangles in this manner, we could draw a regular polygon of fourteen, and then one of seven sides.



However, this can not be done by means only of straight lines and arcs of circles.

The regular polygon of seventeen sides has been drawn in more than one way, using only straight lines and arcs of circles. It has also been shown, that by the same means a regular polygon of two hundred and fifty-seven sides may be drawn. No others are known where the number of the sides is a prime number.

462. Problem.—*Given a regular polygon inscribed in a circle, to circumscribe a similar polygon.*

The vertices of the given polygon divide the circumference into equal parts. Through these points draw tangents. These tangents produced till they meet, form the required polygon (448).

EXERCISES.

463.—1. First in right angles, and then in degrees, express the value of an angle of each regular polygon, from three sides up to twenty.

2. First in right angles, and then in degrees, express the value of an angle at the center, subtended by one side of each of the same polygons.

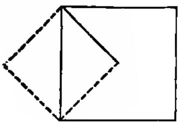
3. To construct a regular octagon of a given side.

4. To circumscribe a circle about a regular polygon.

5. To inscribe a circle in a regular polygon.

6. Given a regular inscribed polygon, to circumscribe a similar polygon whose sides are parallel to the former.

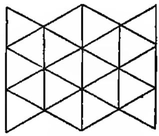
7. The diagonal of a square is to its side as the square root of 2 is to 1.



A PLANE OF REGULAR POLYGONS.

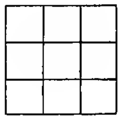
464. In order that any plane surface may be entirely covered by equal polygons, it is necessary that the figures be such, and such only, that the sum of three or more of their angles is equal to four right angles (92).

Hence, to find what regular polygons will fit together so as to cover any plane surface, take them in order according to the number of their sides.



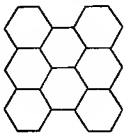
Each angle of an equilateral triangle is equal to one-third of two right angles. Therefore, six such angles exactly make up four right angles; and the equilateral triangle is such a figure as is required.

465. Each angle of the square is a right angle, four of which make four right angles. So that a plane can be covered by equal squares.



One angle of a regular pentagon is the fifth part of six right angles. Three of these are less than, and four exceed four right angles; so that the regular pentagon is not such a figure as is required.

466. Each angle of a regular hexagon is one-sixth of eight right angles. Three such make up four right angles. Hence, a plane may be covered with equal regular hexagons. This combination is remarkable as being the one adopted by bees in forming the honeycomb.



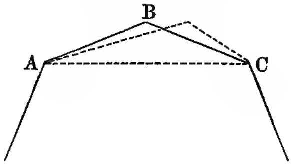
467. Since each angle of a regular polygon evidently increases when the number of sides increases, and since three angles of a regular hexagon are equal to four right angles, therefore, three angles of any regular polygon of more than six sides, must exceed four right angles.

Hence, no other regular figures exist for the purpose here required, except the equilateral triangle, the square, and the regular hexagon.

ISOPERIMETRY.

468. Theorem.—*Of all equivalent polygons of the same number of sides, the one having the least perimeter is regular.*

Of several equivalent polygons, suppose AB and BC to be two adjacent sides of the one having the least perimeter. It is to be proved, first, that these sides are equal.

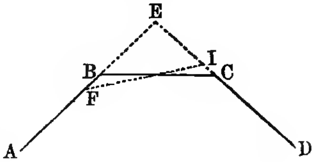


Join AC. Now, if AB and BC were not equal, there could be constructed on the base AC an isosceles triangle equivalent to ABC, whose sides would have less extent (395). Then, this new triangle, with the rest of the polygon, would be equivalent to the given polygon, and have a less perimeter, which is contrary to the hypothesis.

It follows that AB and BC must be equal. So of every two adjacent sides. Therefore, the polygon is equilateral.

It remains to be proved that the polygon will have all its angles equal.

Suppose AB, BC, and CD to be adjacent sides. Produce AB and CD till they meet at E. Now the triangle BCE is isosceles. For if EC, for example, were longer than EB, we could then take EI equal to EB, and EF equal to EC, and we could join FI, making the two triangles EBC and EIF equal (284).



Then, the new polygon, having AFID for part of its perimeter, would be equivalent and isoperimetrical to the given polygon having ABCD as part of its perimeter. But the given polygon has, by hypothesis, the least possible perimeter, and, as just proved, its sides AB, BC, and CD are equal.

If the new polygon has the same area and perimeter, its sides also, for the same reason, must be equal; that is, AF, FI, and ID. But this is absurd, for AF is less than AB, and ID is greater than CD. Therefore, the supposition that EC is greater than EB, which supposition led to this conclusion, is false. Hence, EB and EC must be equal.

Therefore, the angles EBC and ECB are equal (268), and their supplements ABC and BCD are equal. Thus, it may be shown that every two adjacent angles are equal.

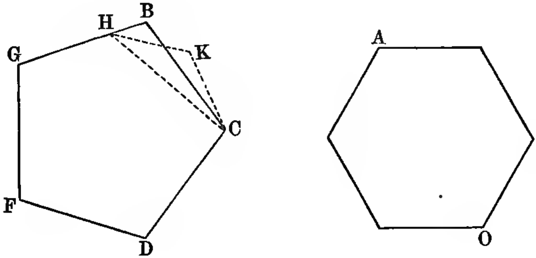
It being proved that the polygon has its sides equal and its angles equal, it is regular.

469. Corollary.—Of all isoperimetrical polygons of the same number of sides, that which is regular has the greatest area.

470. Theorem.—*Of all regular equivalent polygons, that which has the greatest number of sides has the least perimeter.*

It will be sufficient to demonstrate the principle, when one of the equivalent polygons has one side more than the other.

In the polygon having the less number of sides, join the vertex C to any point, as H, of the side BG. Then,



on CH construct an isosceles triangle, CKH, equivalent to CBH.

Then HK and KC are less than HB and BC; therefore, the perimeter GHKCDF is less than the perimeter of its equivalent polygon GBCDF. But the perimeter of the regular polygon AO is less than the perimeter of its equivalent irregular polygon of the same number of sides, GHKCDF (468). So much more is it less than the perimeter of GBCDF.

471. Corollary.—Of two regular isoperimetrical polygons, the greater is that which has the greater number of sides.

EXERCISES.

472.—1. Find the ratios between the side, the radius, and the apothem, of the regular polygons of three, four, five, six, and eight sides.

2. If from any point within a given regular polygon, perpendiculars be let fall on all the sides, the sum of these perpendiculars is a constant quantity.

3. If from all the vertices of a regular polygon, perpendiculars be let fall on a straight line which passes through its center, the

sum of the perpendiculars on one side of this line is equal to the sum of those on the other.

4. If a regular pentagon, hexagon, and decagon be inscribed in a circle, a triangle having its sides respectively equal to the sides of these three polygons will be right angled.

5. If two diagonals of a regular pentagon cut each other, each is divided in extreme and mean ratio.

6. Three houses are built with walls of the same aggregate length; the first in the shape of a square, the second of a rectangle, and the third of a regular octagon. Which has the greatest amount of room, and which the least?

7. Of all triangles having two sides respectively equal to two given lines, the greatest is that where the angle included between the given sides is a right angle.

8. In order to cover a pavement with equal blocks, in the shape of regular polygons of a given area, of what shape must they be that the entire extent of the lines between the blocks shall be a minimum.

9. All the diagonals being formed in a regular pentagon, the figure inclosed by them is a regular pentagon.

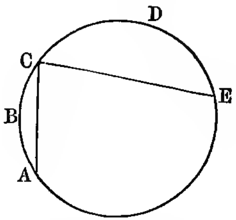
CHAPTER VIII.

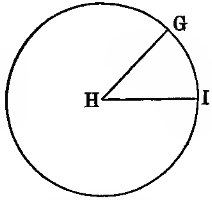
CIRCLES.

473. The properties of the curve which bounds a circle, and of some straight lines connected with it, were discussed in a former chapter. Having now learned the properties of polygons, or rectilinear figures inclosing a plane surface, the student is prepared for the study of the circle as a figure inclosing a surface.

The circle is the only curvilinear figure treated of in Elementary Geometry. Its discussion will complete this portion of the work. The properties of other curves, such as the ellipse which is the figure of the orbits of the planets, are usually investigated by the application of algebra to geometry.

474. A SEGMENT of a circle is that portion cut off by a secant or a chord. Thus, ABC and CDE are segments.





A SECTOR of a circle is that portion included between two radii and the arc intercepted by them. Thus, GHI is a sector.

THE LIMIT OF INSCRIBED POLYGONS.

475. Theorem.—*A circle is the limit of the polygons which can be inscribed in it, also of those which can be circumscribed about it.*

Having a polygon inscribed in a circle, a second polygon may be inscribed of double the number of sides. Then, a polygon of double the number of sides of the second may be inscribed, and the process repeated at will.

Let the student draw a diagram, beginning with an inscribed square or equilateral triangle. Very soon the many sides of the polygon become confused with the circumference. Suppose we begin with a circumscribed regular polygon; here, also, we may circumscribe a regular polygon of double the number of sides. By repeating the process a few times, the polygon becomes inseparable from the circumference.

The mental process is not subject to the same limits that we meet with in drawing the diagrams. We may conceive the number of sides to go on increasing to any number whatever. At each step the inscribed polygon grows larger and the circumscribed grows smaller, both becoming more nearly identical with the circle.

Now, it is evident that by the process described, the polygons can be made to approach as nearly as we please to equality with the circle (35 and 36), but can never entirely reach it. The circle is therefore the limit of the polygons (198).

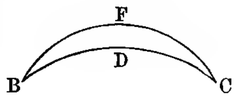
476. Corollary.—A circle is the limit of all regular polygons whose radii are equal to its radius. It is also the limit of all regular polygons whose apothems are equal to its radius. The circumference is the limit of the perimeters of those polygons.

477. By the method of infinites, the circle is considered as a regular polygon of an infinite number of sides, each side being an infinitesimal straight line. But the method of limits is preferred in this place, because, strictly speaking, the circle is not a polygon, and the circumference is not a broken line.

The above theorem establishes only this, that whatever is true of all inscribed, or of all circumscribed polygons, is necessarily true of the circle.

478. Theorem.—*A curve is shorter than any other line which joins its ends, and toward which it is convex.*

For the curve BDC is the limit of those broken lines which have their vertices in it. Then, the curve BDC is less than the line BFC (79).



479. Corollary.—The circumference of a circle is shorter than the perimeter of a circumscribed polygon.

480. Corollary.—The circumference of a circle is longer than the perimeter of an inscribed polygon.

This is a corollary of the Axiom of Distance (54).

481. Theorem.—*A circle has a less perimeter than any equivalent polygon.*

For, of equivalent polygons, that has the least perimeter which is regular (468), and has the greatest number of sides (470).

482. Corollary.—A circle has a greater area than any isoperimetrical figure.

CIRCLES SIMILAR.

483. Theorem.—*Circles are similar figures.*

For angles which intercept like parts of a circumference are equal (207 and 224). Hence, whatever lines be made in one circle, homologous lines, making equal angles, may be made in another.

This theorem may be otherwise demonstrated, thus: Inscribed regular polygons of the same number of sides are similar. The number of sides may be increased indefinitely, and the polygons will still be similar at each successive step. The circles being the limits of the polygons, must also be similar.

484. Theorem.—*Two sectors are similar when the angles made by their radii are equal.*

485. Theorem.—*Two segments are similar when the angles which are formed by radii from the ends of their respective arcs are equal.*

These two theorems are demonstrated by completing the circles of which the given figures form parts. Then the given straight lines in one circle are homologous to those in the other; and any angle in one may have its corresponding equal angle in the other, since the circles are similar.

EXERCISE.

486. When the Tyrian Princess stretched the thongs cut from the hide of a bull around the site of Carthage, what course should she have pursued in order to include the greatest extent of territory?

RECTIFICATION OF CIRCUMFERENCE.

487. Theorem.—*The ratio of the circumference to its diameter is a constant quantity.*

Two circumferences are to each other in the ratio of their diameters. For the perimeters of similar regular polygons are in the ratio of homologous lines (435); and the circumference is the limit of the perimeters of regular polygons (476). Then, designating any two circumferences by C and C′, and their diameters by D and D′,

$$C : C' :: D : D'.$$

Hence, by alternation,

$$C : D :: C' : D'.$$

That is, the ratio of a circumference to its diameter is the same as that of any other circumference to its diameter.

488. The ratio of the circumference to the diameter is usually designated by the Greek letter π, the initial of perimeter.

If we can determine this numerical ratio, multiplying any diameter by it will give the circumference, or a straight line of the same extent as the circumference. This is called the *rectification* of that curve.

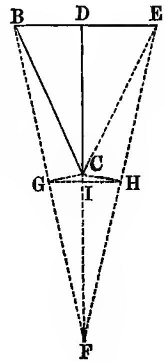
489. The number π is less than 4 and greater than 3. For, if the diameter is 1, the perimeter of the circumscribed square is 4; but this is greater than the circumference (479). And the perimeter of the inscribed regular hexagon is 3, but this is less than the circumference (480).

In order to calculate this number more accurately, let us first establish these two principles:

490. Theorem.—*Given the apothem, radius, and side of a regular polygon; the apothem of a regular polygon of the same length of perimeter, but double the number of sides, is half the sum of the given apothem and radius; and the radius of the polygon of double the number of sides, is a mean proportional between its own apothem and the given radius.*

Let CD be the apothem, CB the radius, and BE the side of a regular polygon. Produce DC to F, making

CF equal to CB. Join BF and EF. From C let the perpendicular CG fall upon BF. Make GH parallel to BE, and join CH and CE.



Now, the triangle BCF being isosceles by construction, the angles CBF and CFB are equal. The sum of these two is equal to the exterior angle BCD (261). Hence, the angle BFD is half the angle BCD. Since DF is, by hypothesis, perpendicular to BE at its center, BCE and BFE are isosceles triangles (108), and the angles BCE and BFE are bisected by the line DF (271). Therefore, the angle BFE is half the angle BCE. That is, the angle BFE is equal to the angle at the center of a regular polygon of double the number of sides of the given polygon (447).

Since GH is parallel to BE,

We have, GH : BE : : GF : BF.

Since GF is the half of BF (271), GH is the half of BE. Then GH is equal to the side of a regular polygon, with the same length of perimeter as the given polygon, and double the number of sides.

Again, FH and FG, being halves of equals, are equal. Also, IF is perpendicular to GH (127). Therefore, we have GH the side, IF the apothem, and GF the radius of the polygon of double the number of sides, with a perimeter equal to that of the given polygon.

Now, the similar triangles give,

FI : FD : : FG : FB.

Therefore, FI is one-half of FD. But FD is, by construction, equal to the sum of CD and CB. Therefore, the apothem of the second polygon is equal to half the sum of the given apothem and radius.

Again, in the right angled triangle GCF (324),

$$FC : FG :: FG : FI.$$

But FC is equal to CB; therefore, FG, the radius of the second polygon, is a mean proportional between the given radius and the apothem of the second.

491. For convenient application of these principles, let us represent the given apothem by a, the radius by r, and the side by s, the apothem of the polygon of double the number of sides by x, and its radius by y.

Then, $\quad x=\frac{a+r}{2}, \quad$ and $x : y :: y : r$.

Hence, $\quad y^2=xr, \quad$ and $y=\sqrt{xr}$.

492. Again, since, in any regular polygon, the apothem, radius, and half the side form a right angled triangle,

We always have, $\quad r^2=a^2+\left(\frac{s}{2}\right)^2$

Hence, $\quad a=\sqrt{r^2-\frac{s^2}{4}}=\frac{1}{2}\sqrt{4r^2-s^2}.$

493. Problem.—*To find the approximate value of the ratio of the circumference to the diameter of a circle.*

Suppose a regular hexagon whose perimeter is unity. Then its side is $\frac{1}{6}$ or .166667, and its radius is the same (454).

By the formula, $a=\frac{1}{2}\sqrt{4r^2-s^2}$, the apothem is

$$\tfrac{1}{2}\sqrt{\tfrac{4}{36}-\tfrac{1}{36}}=\tfrac{1}{12}\sqrt{3}, \text{ or } .144338.$$

Then, by the formula, $x=\frac{1}{2}(a+r)$, the apothem of the regular polygon of twelve sides, the perimeter being unity, is $\frac{1}{2}(\frac{1}{6}+\frac{1}{12}\sqrt{3})$ or .155502. The radius of the same, by the formula $y=\sqrt{xr}$, is .160988. Proceeding in the same way, the following table may be constructed:

REGULAR POLYGONS WHOSE PERIMETER IS UNITY.

Number of sides.	Apothem.	Radius.
6	.144338	.166667
12	.155502	.160988
24	.158245	.159610
48	.158928	.159269
96	.159098	.159183
192	.159141	.159162
384	.159151	.159157
768	.159154	.159155
1536	.159155	.159155

Now, observe that the numbers in the second column express the ratios of the radius of any circle to the perimeters of the circumscribed regular polygons; and that those in the third column express the ratios of the radius to the perimeters of the inscribed polygons. These ratios gradually approach each other, till they agree for six places of decimals. It is evident that by continuing the table, and calculating the ratios to a greater number of decimal places, this approximation could be made as near as we choose.

But it has been already shown that the circumference is less than the perimeter of the circumscribed, and greater than that of the inscribed polygon. Hence, we conclude, that when the circumference is 1, the radius is .159155, with a near approximation to exactness. The diameter, being double the radius, is .31831.

Therefore,

$$\pi = \frac{1}{.31831} = 3.14159.$$

494. It was shown by *Archimedes*, by methods resembling the above, that the value of π is less than $3\frac{1}{7}$, and greater than $3\frac{10}{71}$. This number, $3\frac{1}{7}$, is in very common use for mechanical purposes. It is too great by about one eight-hundredth of the diameter.

About the year 1640, *Adrian Metius* found the nearer approximation $\frac{355}{113}$, which is true for six places of decimals. It is easily retained in the memory, as it is composed of the first three odd numbers, in pairs, 113|355, taking the first three digits for the denominator, and the other three for the numerator.

By the integral calculus, it has been found that π is equal to the series $4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} +$, etc.

By the calculus also, other and shorter methods have been discovered for finding the approximate value of π. In 1853, *Mr. Rutherford* presented to the Royal Society of London a calculation of the value of π to five hundred and thirty decimals, made by *Mr. W. Shanks*, of Houghton-le-Spring.

The first thirty-nine decimals are,

3.141 592 653 589 793 238 462 643 383 279 502 884 197.

EXERCISES.

495.—1. Two wheels, whose diameters are twelve and eighteen inches, are connected by a belt, so that the rotation of one causes that of the other. The smaller makes twenty-four rotations in a minute; what is the velocity of the larger wheel?

2. Two wheels, whose diameters are twelve and eighteen inches, are fixed on the same axle, so that they turn together. A point on the rim of the smaller moves at the rate of six feet per second; what is the velocity of a point on the rim of the larger wheel?

3. If the radius of a car-wheel is thirteen inches, how many revolutions does it make in traveling one mile?

4. If the equatorial diameter of the earth is 7924 miles, what is the length of one degree of longitude on the equator?

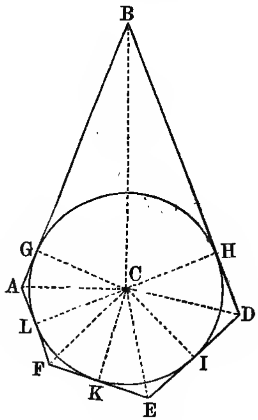
QUADRATURE OF CIRCLE.

496. The *quadrature* or *squaring of the circle*, that is, the finding an equivalent rectilinear figure, is a problem which excited the attention of mathematicians during many ages, until it was demonstrated that it could only be solved approximately.

The solution depends, indeed, on the rectification of the circumference, and upon the following

497. Theorem.—*The area of any polygon in which a circle can be inscribed, is measured by half the product of its perimeter by the radius of the inscribed circle.*

From the center C of the circle, let straight lines extend to all the vertices of the polygon ABDEF, also to all the points of tangency, G, H, I, K, and L.



The lines extending to the points of tangency are radii of the circle, and are therefore perpendicular to the sides of the polygon, which are tangents of the circle (183). The polygon is divided by the lines extending to the vertices into as many triangles as it has sides, ACB, BCD, etc. Regarding the sides of the polygon, AB, BD, etc., as the bases of these several triangles, they all have equal altitudes, for the radii are perpendicular to the sides of the polygon. Now, the area of each triangle is measured by half the product of its base by the common altitude. But the area of the polygon is the sum of the areas of the triangles, and the perimeter of the polygon is the sum of their bases. It follows that the area of the polygon is measured by half the product of the perimeter by the common altitude, which is the radius.

498. Corollary.—The area of a regular polygon is measured by half the product of its perimeter by its apothem.

499. Theorem.—*The area of a circle is measured by half the product of its circumference by its radius.*

For the circle is the limit of all the polygons that may be circumscribed about it, and its circumference is the limit of their perimeters.

500. Theorem.—*The area of a circle is equal to the square of its radius, multiplied by the ratio of the circumference to the diameter.*

For, let r represent the radius. Then, the diameter is $2r$, and the circumference is $\pi \times 2r$, and the area is $\frac{1}{2}\pi \times 2r \times r$, or πr^2 (499); that is, the square of the radius multiplied by the ratio of the circumference to the diameter.

501. Corollary.—The areas of two circles are to each other as the squares of their radii; or, as the squares of their diameters.

502. Corollary.—When the radius is unity, the area is expressed by π.

503. Theorem.—*The area of a sector is measured by half the product of its arc by its radius.*

For, the sector is to the circle as its arc is to the circumference. This may be proved in the same manner as the proportionality of arcs and angles at the center (197 or 202).

504. Since that which is true of every polygon may be shown, by the method of limits, to be true also of plane figures bounded by curves, it follows that in any two similar plane surfaces the ratio of the areas is the second power of the linear ratio.

505. Some of the following exercises are only arithmetical applications of geometrical principles.

The algebraic method may be used to great advantage in many exercises, but every principle or solution that is found in this way, should also be demonstrated by geometrical reasoning.

EXERCISES.

506.—1. What is the length of the radius when the arc of 80° is 10 feet?

2. What is the value, in degrees, of the angle at the center, whose arc has the same length as the radius?

3. What is the area of the segment, whose arc is 60°, and radius 1 foot?

4. To divide a circle into two or more equivalent parts by concentric circumferences.

5. One-tenth of a circular field, of one acre, is in a walk extending round the whole; required the width of the walk.

6. Two irregular garden-plats, of the same shape, contain, respectively, 18 and 32 square yards; required their linear ratio.

7. To describe a circle equivalent to two given circles.

507. The following exercises may require the student to review the leading principles of Plane Geometry.

1. From two points, one on each side of a given straight line, to draw lines making an angle that is bisected by the given line.

2. If two straight lines are not parallel, the difference between the alternate angles formed by any secant, is constant.

3. To draw the minimum tangent from a given straight line to a given circumference.

4. How many circles can be made tangent to three given straight lines?

5. Of all triangles on the same base, and having the same vertical angle, the isosceles has the greatest area.

6. To describe a circumference through a given point, and touching a given line at a given point.

7. To describe a circumference through two given points, and touching a given straight line.

8. To describe a circumference through a given point, and touching two given straight lines.

9. About a given circle to describe a triangle similar to a given triangle.

10. To draw lines having the ratios $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc.

11. To construct a triangle with angles in the ratio 1, 2, 3.

12. Can two unequal triangles have a side and two angles in the one equal to a side and two angles in the other?

13. To construct a triangle when the three lines extending from the vertices to the centers of the opposite sides are given?

14. If two circles touch each other, any two straight lines extending through the point of contact will be cut proportionally by the circumferences.

15 If any point on the circumference of a circle circumscribing an equilateral triangle, be joined by straight lines to the several vertices, the middle one of these lines is equivalent to the other two.

16. Making two diagonals in any quadrilateral, the triangles formed by one have their areas in the ratio of the parts of the other.

17. To bisect any quadrilateral by a line from a given vertex.

18. In the triangle ABC, the side $AB = 13$, $BC = 15$, the altitude $= 12$; required the base AC.

19. The sides of a triangle have the ratio of 65, 70, and 75; its area is 21 square inches; required the length of each side.

20. To inscribe a square in a given segment of a circle.

21. If any point within a parallelogram be joined to each of the four vertices, two opposite triangles, thus formed, are together equivalent to half the parallelogram.

22. To divide a straight line into two such parts that the rectangle contained by them shall be a maximum.

23. The area of a triangle which has one angle of 30°, is one-fourth the product of the two sides containing that angle.

24. To construct a right angled triangle when the area and hypotenuse are given.

25. Draw a right angle by means of Article 413.

26. To describe four equal circles, touching each other exteriorly, and all touching a given circumference interiorly.

27. A chord is 8 inches, and the altitude of its segment 3 inches; required the area of the circle.

28. What is the area of the segment whose arc is 36°, and chord 6 inches?

29. The lines which bisect the angles formed by producing the sides of an inscribed quadrilateral, are perpendicular to each other.

30. If a circle be described about any triangle ABC, then, taking BC as a base, the side AC is to the altitude of the triangle as the diameter of the circle is to the side AB.

31. By the proportion just stated, show that the area of a triangle is measured by the product of the three sides multiplied together, divided by four times the radius of the circumscribing circle.

32. In a quadrilateral inscribed in a circle, the sum of the two rectangles contained by opposite sides, is equivalent to the rectangle contained by the diagonals. This is known as the *Ptolemaic Theorem.*

33. Twice the square of the straight line which joins the vertex of a triangle to the center of the base, added to twice the square of half the base, is equivalent to the sum of the squares of the other two sides.

34. The sum of the squares of the sides of any quadrilateral is equivalent to the sum of the squares of the diagonals, increased by four times the square of the line joining the centers of the diagonals.

35. If, from any point in a circumference, perpendiculars be let fall on the sides of an inscribed triangle, the three points of intersection will be in the same straight line.

GEOMETRY OF SPACE.

CHAPTER IX.

STRAIGHT LINES AND PLANES.

508. The elementary principles of those geometrical figures which lie in one plane, furnish a basis for the investigation of the properties of those figures which do not lie altogether in one plane.

We will first examine those straight figures which do not inclose a space; after these, certain solids, or inclosed portions of space.

The student should bear in mind that when straight lines and planes are given by position merely, without mentioning their extent, it is understood that the extent is unlimited.

LINES IN SPACE.

509. Theorem.—*Through a given point in space there can be only one line parallel to a given straight line.*

This theorem depends upon Articles 49 and 117, and includes Article 119.

510. Theorem.—*Two straight lines in space parallel to a third, are parallel to each other.*

This is an immediate consequence of the definition of parallel lines, and includes Article 118.

511. Problem.—*There may be in space any number of straight lines, each perpendicular to a given straight line at one point of it.*

For we may suppose that while one of two perpendicular lines remains fixed as an axis, the other revolves around it, remaining all the while perpendicular (48). The second line can thus take any number of positions.

This does not conflict with Article 103, for, in this case, the axis is not in the same plane with any two of the perpendiculars.

EXERCISES.

512.—1. Designate two lines which are everywhere equally distant, but which are not parallel.

2. Designate two straight lines which are not parallel, and yet can not meet.

3. Designate four points which do not lie all in one plane.

PLANE AND LINES.

513. Theorem.—*The position of a plane is determined by any plane figure except a straight line.*

This is a corollary of Article 60.

Hence, we say, the plane of an angle, of a circumference, etc.

514. Theorem.—*A straight line and a plane can have only one common point, unless the line lies wholly in the plane.*

This is a corollary of Article 58.

515. When a line and a plane have only one common point, the line is said to *pierce* the plane, and the plane to *cut* the line. The common point is called the *foot* of the line in the plane.

When a line lies wholly in a plane, the plane is said to *pass through* the line.

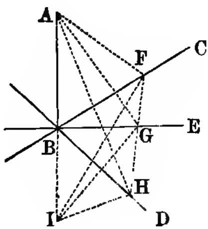
516. Theorem.—*The intersection of two planes is a straight line.*

For two planes can not have three points common, unless those points are all in one straight line (59).

PERPENDICULAR LINES.

517. Theorem.—*A straight line which is perpendicular to each of two straight lines at their point of intersection, is perpendicular to every other straight line which lies in the plane of the two, and passes through their point of intersection.*

In the diagram, suppose D, B, and C to be on the plane of the paper, the point A being above, and I below that plane.



If the line AB is perpendicular to BC and to BD, it is also perpendicular to every other line lying in the plane of DBC, and passing through the point B; as, for example, BE.

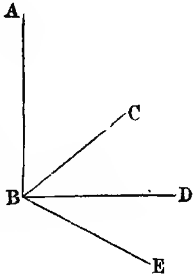
Produce AB, making BI equal to BA, and let any line, as FH, cut the lines BC, BE, and BD, in F, G, and H. Then join AF, AG, AH, and IF, IG, and IH.

Now, since BC and BD are perpendicular to AI at its center, the triangles AFH and IFH have AF equal to IF (108), AH equal to IH, and FH common. Therefore, they are equal, and the angle AHF is equal to IHF. Then the triangles AHG and IHG are equal (284), and the lines AG and IG are equal. Therefore, the line BG, having two points each equally distant from A and I, is perpendicular to the line AI at its center B (109).

In the same way, prove that any other line through B, in the plane of DBC, is perpendicular to AB.

518. Theorem.—*Conversely, if several straight lines are each perpendicular to a given line at the same point, then these several lines all lie in one plane.*

Thus, if BA is perpendicular to BC, to BD, and to BE, then these three all lie in one plane.



BD, for instance, must be in the plane CBE. For the intersection of the plane of ABD with the plane of CBE is a straight line (516). This straight intersection is perpendicular to AB at the point B (517). Therefore, it coincides with BD (103). Thus it may be shown that any other line, perpendicular to AB at the point B, is in the plane of C, B, D, and E.

519. A straight line is said to be *perpendicular to a plane* when it is perpendicular to every straight line which passes through its foot in that plane, and the plane is said to be *perpendicular to the line.* Every line not perpendicular to a plane which cuts it, is called *oblique.*

520. Corollary.—If a plane cuts a line perpendicularly at the middle point of the line, then every point of the plane is equally distant from the two ends of the line (108).

521. Corollary.—If one of two perpendicular lines revolves about the other, the revolving line describes a plane which is perpendicular to the axis.

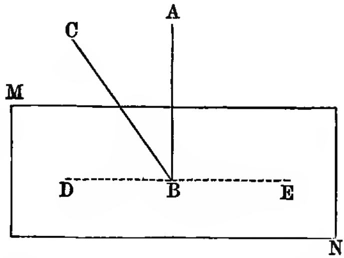
522. Corollary.—Through one point of a straight line there can be only one plane perpendicular to that line.

523. Theorem.—*Through a point out of a plane there can be only one straight line perpendicular to the plane.*

For, if there could be two perpendiculars, then each would be perpendicular to the line in the plane which joins their feet (519). But this is impossible (103).

524. Theorem.—*Through a point in a plane there can be only one straight line perpendicular to the plane.*

Let BA be perpendicular to the plane MN at the point B. Then any other line, BC for example, will be oblique to the plane MN.



For, if the plane of ABC be produced, its intersection with the plane MN will be a straight line.

Let DE be this intersection. Then AB is perpendicular to DE. Hence, BC, being in the plane of A, D, and E, is not perpendicular to DE (103). Therefore, it is not perpendicular to the plane MN (519).

525. Corollary.—The direction of a straight line in space is fixed by the fact that it is perpendicular to a given plane.

The directions of a plane are fixed by the fact that it is perpendicular to a given line.

526. Corollary.—All straight lines which are perpendicular to the same plane, have the same direction; that is, they are parallel to each other.

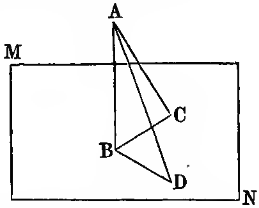
527. Corollary.—If one of two parallel lines is perpendicular to a plane, the other is also.

528. The AXIS of a circle is the straight line perpendicular to the plane of the circle at its center.

OBLIQUE LINES AND PLANES.

529. Theorem.—*If from a point without a plane, a perpendicular and oblique lines be extended to the plane, then two oblique lines which meet the plane at equal distances from the foot of the perpendicular, are equal.*

Let AB be perpendicular, and AC and AD oblique to the plane MN, and the distances BC and BD equal.



Then the triangles ABC and ABD are equal (284), and AC is equal to AD.

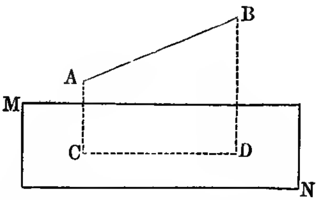
530. Corollary.—A perpendicular is the shortest line from a point to a plane. Hence, the distance from a point to a plane is measured by a perpendicular line.

531. Corollary.—All points of the circumference of a circle are equidistant from any point of its axis.

532. If from all points of a line perpendiculars be let fall upon a plane, the line thus described upon the plane is the *projection* of the given line upon the given plane.

533. Theorem.—*The projection of a straight line upon a plane is a straight line.*

Let AB be the given line, and MN the given plane. Then, from the points A and B, let the perpendiculars, AC and BD, fall upon the plane MN. Join CD.



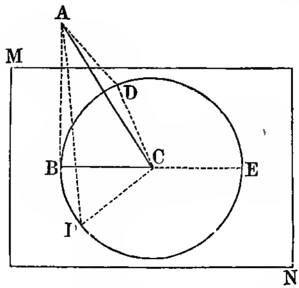
AC and BD, being perpendicular to the same plane, are parallel (526), and lie in one plane (121). Now, every perpendicular to MN let fall from a point of AB, must be parallel to BD, and must therefore lie in the plane AD, and meet the plane MN in some point of CD. Hence, the straight line CD is the projection of the straight line AB on the plane MN.

There is one exception to this proposition. When the given line is perpendicular to the plane, its projection is a point.

534. Corollary.—A straight line and its projection on a plane, both lie in one plane.

535. Theorem.—*The angle which a straight line makes with its projection on a plane, is smaller than the angle it makes with any other line in the plane.*

Let AC be the given line, and BC its projection on the plane MN. Then the angle ACB is less than the angle made by AC with any other line in the plane, as CD.



With C as a center and BC as a radius, describe a circumference in the plane MN, cutting CD at D.

Then the triangles ACD and ACB have two sides of the one respectively equal to two sides of the other. But the third side AD is longer than the third side AB (530). Therefore, the angle ACD is greater than the angle ACB (294).

536. Corollary.—The angle ACE, which a line makes with its projection produced, is larger than the angle made with any other line in the plane.

537. The angle which a line makes with its projec tion in a plane, is called the *Angle of Inclination* of the line and the plane.

PARALLEL LINES AND PLANE.

538. Theorem.—*If a straight line in a plane is parallel to a straight line not in the plane, then the second line and the plane can not have a common point.*

For if any line is parallel to a given line in a plane, and passes through any point of the plane, it will lie wholly in the plane (121). But, by hypothesis, the second line does not lie wholly in the plane. Therefore, it can not pass through any point of the plane, to whatever extent the two may be produced.

539. Such a line and plane, having the same direction, are called *parallel.*

540. Corollary.—If one of two parallel lines is parallel to a plane, the other is also.

541. Corollary.—A line which is parallel to a plane is parallel to its projection on that plane.

542. Corollary.—A line parallel to a plane is everywhere equally distant from it.

APPLICATIONS.

543. Three points, however placed, must always be in the same plane. It is on this principle that stability is more readily obtained by three supports than by a greater number. A three-legged stool *must* be steady, but if there be four legs, their ends should be in one plane, and the floor should be level. Many surveying and astronomical instruments are made with three legs.

544. The use of lines perpendicular to planes is very frequent in the mechanic arts. A ready way of constructing a line perpendicular to a plane is by the use of two squares (114). Place the angle of each at the foot of the desired perpendicular, one side of each square resting on the plane surface. Bring their perpendicular sides together. Their position must then be that of a perpendicular to the plane, for it is perpendicular to two lines in the plane.

545. When a circle revolves round its axis, the figure undergoes no real change of position, each point of the circumference taking successively the position deserted by another point.

On this principle is founded the operation of millstones. Two circular stones are placed so as to have the same axis, to which their faces are perpendicular, being, therefore, parallel to each other. The lower stone is fixed, while the upper one is made to revolve. The relative position of the faces of the stones undergoes no change during the revolution, and their distance being properly regulated, all the grain which passes between them will be ground with the same degree of fineness.

546. In the turning lathe, the axis round which the body to be turned is made to revolve, is the axis of the circles formed by the cutting tool, which removes the matter projecting beyond a proper distance from the axis. The cross section of every part of the thing turned is a circle, all the circles having the same axis.

DIEDRAL ANGLES.

547. A Diedral Angle is formed by two planes meeting at a common line. This figure is also called simply a *diedral.* The planes are its *faces,* and the intersection is its *edge.*

In naming a diedral, four letters are used, one in each face, and two on the edge, the letters on the edge being between the other two.

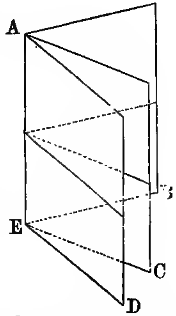
This figure is called a diedral *angle,* because it is similar in many respects to an angle formed by two lines.

MEASURE OF DIEDRALS.

548. The quantity of a diedral, as is the case with a linear angle, depends on the difference in the directions of the faces from the edge, without regard to the extent of the planes. Hence, two diedrals are equal when they can be so placed that their planes will coincide.

549. Problem.—*One diedral may be added to another.*

In the diagram, AB, AC, and AD represent three planes having the common intersection AE.



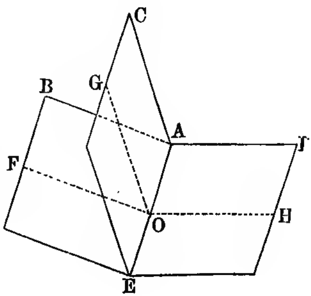
Evidently the sum of BEAC and CEAD is equal to BEAD.

550. Corollary. — Diedrals may be subtracted one from another. A diedral may be bisected or divided in any required ratio by a plane passing through its edge.

551. But there are in each of these planes any number of directions. Hence, it is necessary to determine which of these is properly the direction of the face from the edge. For this purpose, let us first establish the following principle:

552. Theorem.—*One diedral is to another as the plane angle, formed in the first by a line in each face perpendicular to the edge, is to the similarly formed angle in the other.*

Thus, if FO, GO, and HO are each perpendicular to AE, then the diedral CEAD is to the diedral BEAD as the angle GOH is to the angle FOH. This may be demonstrated in the same manner as the proposition in Article 197.



553. Corollary.—A diedral is said to be measured by the plane angle formed by a line in each of its faces perpendicular to the edge.

554. Corollary.—Accordingly, a diedral angle may be acute, obtuse, or right. In the last case, the planes are perpendicular to each other.

555. Many of the principles of plane angles may be applied to diedrals, without further demonstration.

All right diedral angles are equal (90).

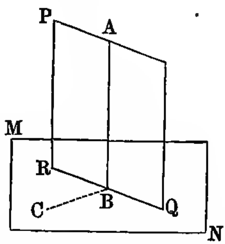
When the sum of several diedrals is measured by two right angles, the outer faces form one plane (100).

When two planes cut each other, the opposite or vertical diedrals are equal (99).

PERPENDICULAR PLANES.

556. Theorem.—*If a line is perpendicular to a plane, then any plane passing through this line is perpendicular to the other plane.*

If AB in the plane PQ is perpendicular to the plane MN, then AB must be perpendicular to every line in MN which passes through the point B (519); that is, to RQ, the intersection of the two planes, and to BC, which is made perpendicular to the intersection RQ. Then, the angle ABC measures the inclination of the two planes (553), and is a right angle. Therefore, the planes are perpendicular.



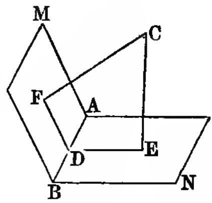
557. Corollary.—Conversely, if a plane is perpendicular to another, a straight line, which is perpendicular to one of them, at some point of their intersection, must lie wholly in the other plane (524).

558. Corollary.—If two planes are perpendicular to a third, then the intersection of the first two is a line perpendicular to the third plane.

OBLIQUE PLANES.

559. Theorem.—*If from a point within a diedral, perpendicular lines be made to the two faces, the angle of these lines is supplementary to the angle which measures the diedral.*

Let M and N be two planes whose intersection is AB, and CF and CE perpendiculars let fall upon them from the point C; and DF and DE the intersections of the plane FCE with the two planes M and N. Then the plane FCE must be perpendicular to each of the planes M and N (556).



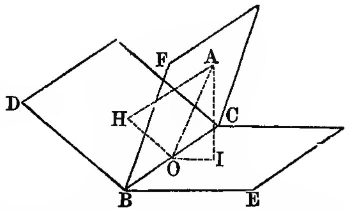
Hence, the line AB is perpendicular to the plane FCE (558), and the angles ADF and ADE are right angles. Then the angle FDE measures the diedral. But in the quadrilateral FDEC, the two angles F and E are right angles. Therefore, the other two angles at C and D are supplementary.

560. Theorem.—*Every point of a plane which bisects a diedral is equally distant from its two faces.*

Let the plane FC bisect the diedral DBCE. Then it is to be proved that every point of this plane, as A, for example, is equally distant from the planes DC and EC.

From A let the perpendiculars AH and AI fall upon the faces DC and EC, and let IO, AO, and HO be the intersections of the plane of the angle IAH with the three given planes.

Then it may be shown, as in the last theorem, that the angle HOA measures the diedral FBCD, and the angle IOA the diedral FBCE. But these diedrals are equal, by hypothesis. Therefore, the line AO bisects the angle IOH,



and the point A is equally distant from the lines OH and OI (113). But the distance of A from these lines is measured by the same perpendiculars, AH and AI, which measure its distance from the two faces DC and EC. Therefore, any point of the bisecting plane is equally distant from the two faces of the given diedral.

APPLICATIONS.

561. Articles 548 to 554 are illustrated by a door turning on its hinges. In every position it is perpendicular to the floor and ceiling. As it turns, it changes its inclination to the wall, in which it is constructed, the angle of inclination being that which is formed by the upper edge of the door and the lintel.

562. The theory of diedrals is as important in the study of magnitudes bounded by planes, as is the theory of angles in the study of polygons.

This is most striking in the science of crystallography, which teaches us how to classify mineral substances according to their geometrical forms. Crystals of one kind have edges of which the diedral angles measure a certain number of degrees, and crystals of another kind have edges of a different number of degrees. Crystals of many species may be thus classified, by measuring their diedrals.

563. The plane of the surface of a liquid at rest is called *horizontal*, or the plane of the horizon. The direction of a plumbline when the weight is at rest, is a *vertical line.* The vertical line is perpendicular to the horizon, the positions of both being governed by the same causes. Every line in the plane of the horizon, or parallel to it, is called a horizontal line, and every plane passing through a vertical line is called a *vertical plane.* Every vertical plane is perpendicular to the horizon.

Horizontal and vertical planes are in most frequent use. Floors, ceilings, etc., are examples of the former, and walls of the latter. The methods of using the builder's level and plummet to determine the position of these, are among the simplest applications of geometrical principles.

Civil engineers have constantly to observe and calculate the position of horizontal and vertical planes, as all objects are referred to these. The astronomer and the navigator, at every step, refer to the horizon, or to a vertical plane.

EXERCISES.

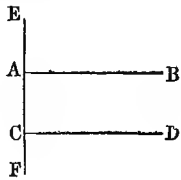
564.—1. If, from a point without a plane, several equal oblique lines extend to it, they make equal angles with the plane.

2. If a line is perpendicular to a plane, and if from its foot a perpendicular be let fall on some other line which lies in the plane, then this last line is perpendicular to the plane of the other two.

3. What is the locus of those points in space, each of which is equally distant from two given points?

PARALLEL PLANES.

565. Two planes which are perpendicular to the same straight line, at different points of it, are both fixed in position (525), and they have the same directions. If the parallel lines AB and CD revolve about the line EF, to which they are both perpendicular, then each of the revolving lines describes a plane.



Every direction assumed by one line is the same as that of the other, and, in the course of a complete revolution, they take all the possible directions of the two planes.

Two planes which have the same directions are called *parallel planes*.

PARALLELISM consists in having the same direction, whether it be of two lines, of two planes, or of a line and a plane.

566. Corollary.—Two planes parallel to a third are parallel to each other.

567. Corollary.—Two planes perpendicular to the same straight line are parallel to each other.

568. Corollary.—A straight line perpendicular to one of two parallel planes is perpendicular to the other.

569. Corollary.—Every straight line in one of two parallel planes has its parallel line in the other plane. Therefore, every straight line in one of the planes is parallel to the other plane.

570. Corollary.—Since through any point in a plane there may be a line parallel to any line in the same plane (121), therefore, in one of two parallel planes, and through any point of it, there may be a straight line parallel to any straight line in the other plane.

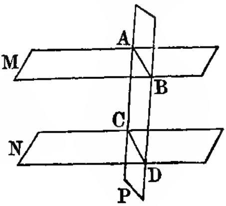
571. Theorem.—*Two parallel planes can not meet.*

For, if they had a common point, being parallel, they would have the same directions from that point, and therefore would coincide throughout, and be only one plane.

572. Theorem.—*The intersections of two parallel planes by a third plane are parallel lines.*

Let AB and CD be the intersections of the two parallel planes M and N, with the plane P.

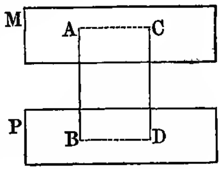
Now, if through C there be a line parallel to AB, it must lie in the plane P (121), and also in the plane N (570). Therefore, it is the intersection CD, and the two intersections are parallel lines.



When two parallel planes are cut by a third plane, eight diedrals are formed, which have properties similar to those of Articles 124 to 128.

573. Theorem.—*The parts of two parallel lines intercepted between parallel planes are equal.*

For, if the lines AB and CD are parallel, they lie in one plane. Then AC and BD are the intersections of this plane with the two parallel planes M and P. Hence, AC is parallel to BD, and AD is a parallelogram. Therefore, AB is equal to the opposite side CD.



574. Theorem.—*Two parallel planes are everywhere equally distant.*

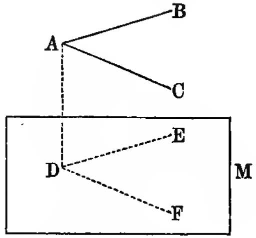
For the shortest distance from any point of one plane to the other, is measured by a perpendicular. But these perpendiculars are all parallel (526), and therefore equal to each other.

575. Theorem.—*If the two sides of an angle are each parallel to a given plane, then the plane of that angle is parallel to the given plane.*

If AB and AC are each parallel to the plane M, then the plane of BAC is parallel to the plane M.

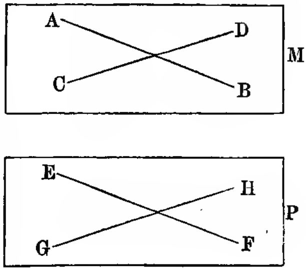
From A let the perpendicular AD fall upon the plane M, and let the projections of AB and AC on the plane M be respectively DE and DF.

Since DE is parallel to AB (541), DA is perpendicular to AB (127). For a like reason, DA is perpendicular to AC. Therefore, DA is perpendicular to the plane of BAC (517), and the two planes being perpendicular to the same line are parallel to each other (567).



576. Theorem.—*If two straight lines which cut each other are respectively parallel to two other straight lines which cut each other, then the plane of the first two is parallel to the plane of the second two.*

Let AB be parallel to EF, and CD parallel to GH. Then the planes M and P are parallel.



For AB being parallel to EF, is parallel to the plane P in which it lies (538). Also, CD is parallel to the plane P, for the same reason. Therefore, the plane M is parallel to the plane P (575).

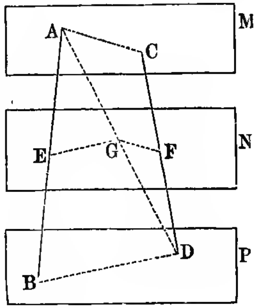
577. Corollary.—The angles made by the first two lines are respectively the same as those made by the second two. For they are the differences between the same directions.

This includes the corresponding principle of Plane Geometry.

578. Theorem.—*Straight lines cut by three parallel planes are divided proportionally.*

If the line AB is cut at the points A, E, and B, and the line CD at the points C, F, and D, by the parallel planes M, N, and P, then

AE : EB : : CF : FD.



Join AC, AD, and BD. AD pierces the plane N in the point G. Join EG and GF.

Now, EG and BD are parallel, being the intersections of the parallel planes N and P by the third plane ABD (572). Hence (313),

AE : EB : : AG : GD.

For a like reason,

AG : GD : : CF : FD.

Therefore, AE : EB : : CF : FD.

APPLICATION.

579. The general problem of *perspective* in drawing, consists in representing upon a plane surface the apparent form of objects in sight. This plane, the plane of the picture, is supposed to be between the eye and the objects to be drawn. Then each object is to be represented upon the plane, at the point where it would be pierced by the visual ray from the object to the eye.

All the visual rays from one straight object, such as the top of a wall, or one corner of a house, lie in one plane (60). Their intersection with the plane of the picture must be a straight line (516). Therefore, all straight objects, whatever their position, must be drawn as straight lines.

Two parallel straight objects, if they are also parallel to the plane of the picture, will remain parallel in the perspective. For the lines drawn must be parallel to the objects (572), and therefore to each other.

Two parallel lines, which are not parallel to the plane of the picture, will meet in the perspective. They will meet, if produced, at that point where the plane of the picture is pierced by a line from the eye parallel to the given lines.

EXERCISES.

580.—1. A straight line makes equal angles with two parallel planes.

2. Two parallel lines make the same angle of inclination with a given plane.

3. The projections of two parallel lines on a plane are parallel.

4. When two planes are each perpendicular to a third, and their intersections with the third plane are parallel lines, then the two planes are parallel to each other.

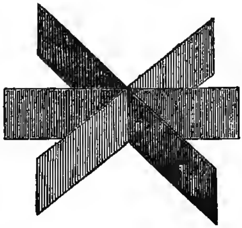
5. If two straight lines be not in the same plane, one straight line, and only one, may be perpendicular to both of them.

6. Demonstrate the last sentence of Article 579.

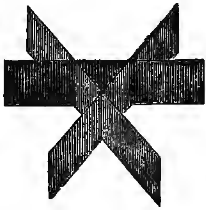
TRIEDRALS.

581. When three planes cut each other, three cases are possible.

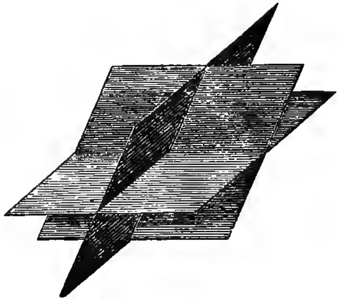
1st. The intersections may coincide. Then six diedrals are formed, having for their common edge the intersection of the three planes.



2d. The three intersections may be parallel lines. Then one plane is parallel to the intersection of the other two.



3d. The three intersections may meet at one point. Then the space about the point is divided by the three planes into eight parts.



The student will apprehend this better when he reflects that two intersecting planes make four diedrals. Now, if a third plane cut through the intersection of the first two, it will divide each of the diedrals into two parts, making eight in all. Each of these parts is called a triedral, because it has three faces.

A fourth case is impossible. For, since any two of the intersections lie in one plane, they must either be parallel, or they meet. If two of the intersections meet, the point of meeting must be common to the three planes, and must therefore be common to all the intersections. Hence, the three intersections either have more than one point common, only one point common, or no point common. But these are the three cases just considered.

582. A Triedral is the figure formed by three planes meeting at one point. The point where the planes and intersections all meet, is called the *vertex* of the triedral. The intersections are its *edges*, and the planes are its *faces*.

The corners of a room, or of a chest, are illustrations of triedrals with rectangular faces. The point of a triangular file, or of a small-sword, has the form of a triedral with acute faces.

The triedral has many things analogous to the plane triangle. It has been called a solid triangle; and more frequently, but with less propriety, a solid angle. The three faces, combined two and two, make three diedrals, and the three intersections, combined two and two, make three plane angles. These six are the six elements or principal parts of a triedral.

Each face is the plane of one of the plane angles, and two faces are said to be equal when these angles are equal.

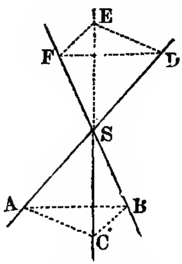
Two triedrals are said to be equal when their several planes may coincide, without regard to the extent of the planes. Since each plane is determined by two lines, it is evident that two triedrals are equal when their several edges respectively coincide.

583. A triedral which has one rectangular diedral, that is, whose measure is a right angle, is called a *rectangular triedral.* If it has two, it is *birectangular;* if it has three, it is *trirectangular.*

A triedral which has two of its faces equal, is called *isosceles;* if all three are equal, it is *equilateral.*

SYMMETRICAL TRIEDRALS.

584. If the edges of a triedral be produced beyond the vertex, they form the edges of a new triedral. The faces of these two triedrals are respectively equal, for the angles are vertical.



Thus, the angles ASC and ESD are equal; also, the angles BSC and FSE are equal, and the angles ASB and DSF.

The diedrals whose edges are FS and BS are also equal, since, being formed by the same planes, EFSBC and DFSBA, they are vertically opposite diedrals (555). The same is true of the diedrals whose edges are DS and SA, and of the diedrals whose edges are ES and SC.

In the diagram, suppose ASB to be the plane of the paper, C being above and E below that plane.

But the two triedrals are not equal, for they can not be made to coincide, although composed of parts which are respectively equal. This will be more evident if the student will imagine himself within the first triedral, his head toward the vertex, and his back to the plane ASB. Then the plane ASC will be on the right hand, and BSC on the left. Then let him imagine himself in the other triedral, his head toward the vertex, and his back to the plane FSD, which is equal to ASB. Then the plane on the right will be FSE, which is equal to BSC, the one that had been on the left; and the plane now on the left will be DSE, equal to the one that had been on the right.

Now, since the equal parts are not similarly situated, the two figures can not coincide.

Then the difference between these two triedrals consists in the opposite order in which the parts are arranged. This may be illustrated by two gloves, which we may suppose to be composed of exactly equal parts. But they are arranged in reverse order. The right hand glove will not fit the left hand. The two hands themselves are examples of the same kind.

585. When two magnitudes are composed of parts respectively equal, but arranged in reverse order, they are said to be *symmetrical magnitudes*.

The word symmetrical, as here used, has essentially the same meaning as that given in Plane Geometry (158). Two symmetrical plane figures, or parts of a figure, are divided by a straight line, while two such figures in space are divided by a plane.

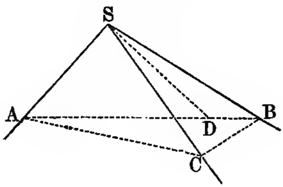
When two plane figures are symmetrical, they are also equal, for one can be turned over to coincide with the other, as with the figures *m* and *n* in Article 282. But this is not possible, as just shown, with figures that are not in one plane.

ANGLES OF A TRIEDRAL.

586. Theorem.—*Each plane angle of a triedral is less than the sum of the other two.*

The theorem is demonstrated, when it is shown that the greatest angle is less than the sum of the other two.

Let ASB be the largest of the three angles of the triedral S. Then, from the angle ASB take the part ASD, equal to the angle ASC. Join the edges SA and SB by any straight line AB. Take SC equal to SD, and join AC and BC.

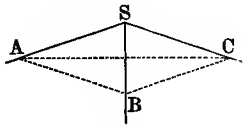


Since the triangles ASD and ASC are equal (284), AD is equal to AC. But AB is less than the sum of AC and BC, and from these, subtracting the equals AD and AC, we have BD less than BC. Hence, the triangles BSD and BSC have two sides of the one equal to two sides of the other, and the third side BD less than the third side BC. Therefore, the included angle BSD is less than the angle BSC. Adding to these the equal angles ASD and ASC, we have the angle ASB less than the sum of the angles ASC and BSC.

587. Theorem.—*The sum of the plane angles which form a triedral is always less than four right angles.*

Through any three points, one in each edge of the triedral, let the plane ABC pass, making the intersections AB, BC, and AC, with the faces.

There is thus formed a triedral at each of the points A, B, and C. Then the angle BAC is less than the sum of BAS and CAS (586). The angle ABC is less than the sum of ABS and CBS. The angle BCA is less than the sum of ACS and BCS. Adding together these inequalities, we find that the sum of the angles of the triangle ABC, which is two right angles, is less than the sum of the six angles at the bases of the triangles on the faces of the triedral S.



The sum of all the angles of these three triangles is six right angles. Therefore, since the sum of those at the bases is more than two right angles, the sum of those at the vertex S must be less than four right angles.

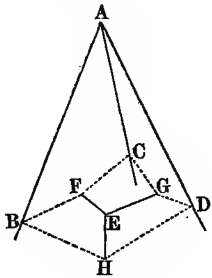
588. To assist the student to understand this theorem, let him take any three points on the paper or blackboard for A, B, and C. Take S at some distance from the surface, so that the plane angles formed at S will be quite acute. Then let S approach the surface of the triangle ABC. Evidently the angles at S become larger and larger, until the point S touches the surface of the triangle, when the sum of the angles becomes four right angles, and, at the same time, the triedral becomes one plane.

SUPPLEMENTARY TRIEDRALS.

589. Theorem.—*If, from a point within a triedral, perpendicular lines fall on the several faces, these lines* *will be the edges of a second triedral, whose faces will be supplements respectively of the diedrals of the first; and the faces of the first will be respectively supplements of the diedrals of the second triedral.*

A plane angle is not strictly the supplement of a diedral, but we understand, by this abridged expression, that the plane angle is the supplement of that which measures the diedral.

If from the point E, within the triedral ABCD, the perpendiculars EF, EG, and EH fall on the several faces, then these lines form a second triedral, whose faces are FEH, FEG, and GEH.



Then the angle FEH is the supplement of the diedral whose edge is BA, for the sides of the angle are perpendicular to the faces of the diedral (559). For the same reason, the angle FEG is the supplement of the diedral whose edge is CA, and the angle GEH is the supplement of the diedral whose edge is DA.

But it may be shown that these two triedrals have a reciprocal relation; that is, that the property just proved of the second toward the first, is also true of the first toward the second.

Let BF and BH be the intersections of the face FEH with the faces BAC and BAD; CF and CG be the intersections of the face FEG with the faces BAC and CAD; and DG and DH be the intersections of the face GEH with the faces CAD and BAD.

Now, since the plane FBH is perpendicular to each of the planes BAC and BAD (556), their intersection AB is perpendicular to the plane FBH (558). For a like reason, AC is perpendicular to the plane FCG and AD is perpendicular to the plane GDH. Then, reasoning as above, we prove that the angle BAC is the supplement of the diedral whose edge is FE; and that each of the other faces of the first triedral is a supplement of a diedral of the second.

590. Two triedrals, in which the faces and diedral angles of the one are respectively the supplements of the diedral angles and faces of the other, are called *supplementary triedrals.*

Instead of placing supplementary triedrals each within the other, as above, they may be supposed to have their vertices at the same point. Thus, at the point A, erect a perpendicular to each of the three faces of the triedral ABCD, and on the side of the face toward the triedral. A second triedral is thus formed, which is supplementary to the triedral ABCD, and is symmetrical to the one formed within.

SUM OF THE DIEDRALS.

591. Theorem.—*In every triedral the sum of the three diedral angles is greater than two right angles, and less than six.*

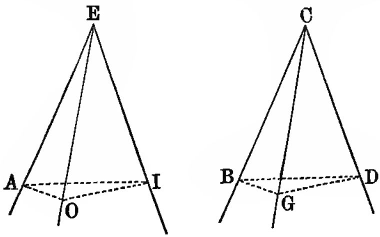
Consider the supplementary triedral, with the given one. Now, the sum of the three diedrals of the given triedral, and of the three faces of its supplementary triedral, must be six right angles; for the sum of each pair is two right angles. But the sum of the faces of the supplementary triedral is less than four right angles (587), and is greater than zero. Subtracting this sum from the former, the remainder, being the sum of the three diedrals of the given triedral, is greater than two and less than six right angles.

EQUALITY OF TRIEDRALS.

592. Theorem.—*When two triedrals have two faces, and the included diedral of the one respectively equal to the corresponding parts of the other, then the remaining face and diedrals of the first are respectively equal to the corresponding parts of the other.*

There are two cases to be considered.

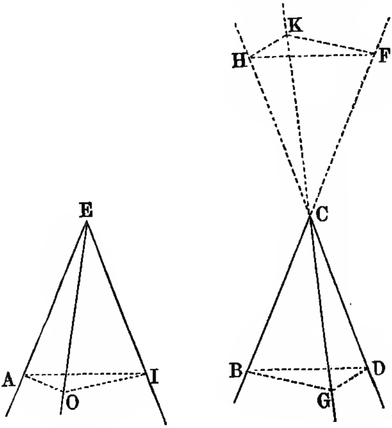
1st. Suppose the angles AEO and BCG equal, and the angles AEI and BCD equal, also the included diedrals whose edges are AE and BC. Let the arrangement be the same in both, so that, if we go around one triedral in the order O, A, I, O, and around the other in the order G, B, D, G, in both cases the triedral will be on the right. Then it may be proved that the two triedrals are equal.



Place the angle BCD directly upon its equal, AEI. Since the diedrals are equal, and are on the same side of the plane AEI, the planes BCG and AEO will coincide. Since the angles BCG and AEO are equal, the lines CG and EO will coincide. Thus, the angles DCG and IEO coincide, and the two triedrals coincide throughout.

2d. Let the angles AEO and DCG be equal, and the angles AEI and BCD, also the included diedrals, whose edges are AE and DC. But let the arrangement be reverse; that is, if we go around one triedral in the order O, A, I, O, and around the other in the order G, D, B, G, in one case the triedral will be to the right, and in the other it will be to the left of us. Then it may be proved that the two triedrals are symmetrical.

One of the triedrals can be made to coincide with the symmetrical of the other; for if the edges BC, GC, and DC be produced beyond C, the triedral CFHK will have two faces and the included diedral respectively equal to those parts of the triedral EAOI, and arranged in the same order; that is, the reverse of the triedral CDGB. Hence, as just shown, the triedrals CFHK and EAOI are equal.



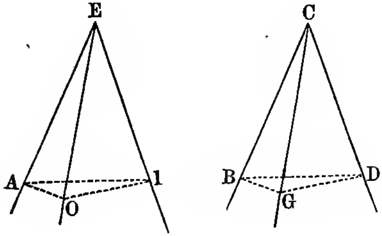
Therefore, EAOI and CDGB are symmetrical triedrals.

In both cases, all the parts of each triedral are respectively equal to those of the other.

593. Theorem.—*When two triedrals have one face and the two adjacent diedrals of the one respectively equal to the corresponding parts of the other, then the remaining faces and diedral of the first are respectively equal to the corresponding parts of the other.*

Suppose that the faces AEI and BCD are equal, that the diedrals whose edges are AE and BC are equal, that the diedrals whose edges are IE and DC are equal, and that these parts are similarly arranged in the two triedrals. Then the one may coincide with the other.

For BCD may coincide with its equal AEI, BC falling on AE. Then the plane of BCG must coincide with that of AEO, since the diedrals are equal; and the line CG will fall in the plane of AEO. For a similar reason CG will fall on the plane of IEO. Therefore, it must coincide with their intersection EO, and the two triedrals coincide throughout.

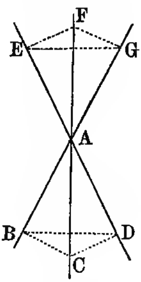


When the equal parts are in reverse order in the two triedrals, the arrangement in one must be the same as in the symmetrical of the other. Therefore, in that case, the two triedrals would be symmetrical.

In both cases, all the parts of each triedral are respectively equal to those of the other.

594. Theorem.—*An isosceles triedral and its symmetrical are equal.*

Let ABCD be an isosceles triedral, having the faces BAC and DAC equal, and let AEFG be its symmetrical triedral.



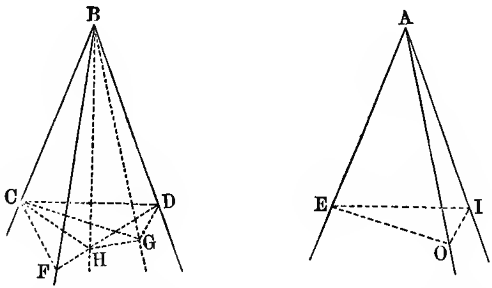
Now, the faces BAC, DAC, FAG, and FAE, are all equal to each other. The diedrals whose edges are AC and AF being vertical, are also equal. Hence, the faces mentioned being all equal, corresponding equal parts may be taken in the same order in both triedrals; that is, the face EAF equal to the face BAC, and the face FAG equal to CAD. Therefore, the two triedrals are equal.

595. Corollary.—In an isosceles triedral, the diedrals opposite the equal faces are equal. For the diedrals whose edges are AB and AD, are each equal to the diedral whose edge is AE.

596. Corollary.— Conversely, if in any triedral two of the diedral angles are equal, then the faces opposite these diedrals are equal, and the triedral is isosceles. For, as in the above theorem, the given triedral can be shown to be equal to its symmetrical.

597. Theorem.—*When two triedrals have two faces of the one respectively equal to two faces of the other, and the included diedrals unequal, then the third faces are unequal, and that face is greater which is opposite the greater diedral.*

Suppose that the faces CBD and EAI are equal, and



that the faces CBF and EAO are also equal, but that the diedral whose edge is CB is greater than the diedral whose edge is EA. Then the face FBD will be greater than the face OAI.

Through the line BC, let a plane GBC pass, making with the plane DBC a diedral equal to that whose edge is AE. In this plane, make the angle CBG equal to EAO. Let the diedral FBCG be bisected by the plane

HBC, BH being the intersection of this plane with the plane FBD.

Then the two triedrals BCDG and AEIO, having two faces and the included diedral in the one equal to the corresponding parts in the other, will have the remaining parts equal. Hence, the faces DBG and IAO are equal.

Again, the two triedrals BCFH and BCGH have the faces CBF and CBG equal, by construction, the face CBH common, and the included diedrals equal, by construction. Therefore, the third faces FBH and GBH are equal.

To each of these equals add the face HBD, and we have the face FBD equal to the sum of GBH and HBD. But in the triedral BDGH, the face DBG is less than the sum of the other two faces, GBH and HBD (586) Hence, the face DBG is less than FBD. Therefore, the face OAI, equal to DBG, is less than FBD.

598. Corollary.—Conversely, when two triedrals have two faces of the one respectively equal to two faces of the other, and the third faces are unequal, then the diedral opposite the greater face is greater than the diedral opposite the less.

599. Theorem.—*When two triedrals have their three faces respectively equal, their diedrals will be respectively equal; and the two triedrals are either equal, or they are symmetrical.*

When two faces of one triedral are respectively equal to those of another, if the included diedrals are unequal, then the opposite faces are unequal (597). But, by the hypothesis of this theorem, the third faces are equal. Therefore, the diedrals opposite to those faces must be equal.

In the same manner, it may be shown that the other diedral angles of the one, are equal to the corresponding diedral angles of the other triedral. Therefore, the triedrals are either equal or symmetrical, according to the arrangement of their parts.

600. Theorem. — *Two triedrals which have their diedrals respectively equal, have also their faces respectively equal; and the two triedrals are either equal, or they are symmetrical.*

Consider the supplementary triedrals of the two given triedrals. These will have their faces respectively equal, because they are the supplements of equal diedral angles (589). Since their faces are equal, their diedrals are equal (599). Then the two given triedrals, having their faces the supplements of these equal diedrals, will have those faces equal; and the triedrals are either equal or symmetrical, according to the arrangement of their parts.

601. The student may notice, in every other case of equal triedrals, the analogy to a case of equality of triangles; but the theorem just demonstrated has nothing analogous in plane geometry.

602. Corollary. — All trirectangular triedrals are equal.

603. Corollary.—In all cases where two triedrals are either equal or supplementary, equal faces are opposite equal diedral angles.

EXERCISES.

604.—1. In any triedral, the greater of two faces is opposite to the greater diedral angle; and conversely.

2. Demonstrate the principles stated in the last sentence of Article 590.

3. If a triedral have one right diedral angle, then an adjacent

face and its opposite diedral are either both acute, both right, or both obtuse.

POLYEDRALS.

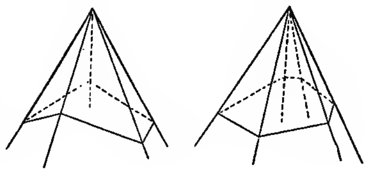
605. A POLYEDRAL is the figure formed by several planes which meet at one point. Thus, a polyedral is composed of several angles having their vertices at a common point, every edge being a side of two of the angular faces. The triedral is a polyedral of three faces.

606. Problem.—*Any polyedral of more than three faces may be divided into triedrals*

For a plane may pass through any two edges which are not adjacent. Thus, a polyedral of four faces may be divided into two triedrals; one of five faces, into three; and so on.

607. This is like the division of a polygon into triangles. The plane passing through two edges not adjacent is called a *diagonal plane.*

A polyedral is called *convex*, when every possible diagonal plane lies within the figure; otherwise it is called *concave.*



608. Corollary.—If the plane of one face of a convex polyedral be produced, it can not cut the polyedral.

609. Corollary.—A plane may pass through the vertex of a convex polyedral, without cutting any face of the polyedral.

610. Corollary.—A plane may cut all the edges of a convex polyedral. The section is a convex polygon.

611. When any figure is cut by a plane, the figure that is defined on the plane by the limits of the figure so cut, is called a *plane section.*

Several properties of triedrals are common to other polyedrals.

612. Theorem.—*The sum of all the angles of a convex polyedral is less than four right angles.*

For, suppose the polyedral to be cut by a plane, then the section is a polygon of as many sides as the polyedral has faces. Let n represent the number of sides of the polygon. The plane cuts off a triangle on each face of the polyedral, making n triangles. Now, the sum of the angles of this polygon is $2n-4$ right angles (424), and the sum of the angles of all these triangles is $2n$ right angles. Let v right angles represent the sum of the angles at the vertex of the polyedral; then, $2n$ right angles being the sum of all the angles of the triangles, $2n-v$ is the sum of the angles at their bases.

Now, at each vertex of the polygon is a triedral having an angle of the polygon for one face, and angles at the bases of the triangles for the other two faces. Then, since two faces of a triedral are greater than the third, the sum of all the angles at the bases of the triangles is greater than the sum of the angles of the polygon. That is,

$$2n-v>2n-4.$$

Adding to both members of this inequality, $v+4$, and subtracting $2n$, we have $4>v$. That is, the sum of the angles at the vertex is less than four right angles.

This demonstration is a generalization of that of Article 587. The student should make a diagram and special demonstration for a polyedral of five or six faces.

613. Theorem.—*In any convex polyedral, the sum of the diedrals is greater than the sum of the angles of a polygon having the same number of sides that the polyedral has faces.*

Let the given polyedral be divided by diagonal planes into triedrals. Then this theorem may be demonstrated like the analogous proposition on polygons (423). The remark made in Article 346 is also applicable here.

DESCRIPTIVE GEOMETRY.

614. In the former part of this work, we have found problems in drawing to be the best exercises on the principles of Plane Geometry. At first it appears impossible to adapt such problems to the Geometry of Space; for a drawing is made on a plane surface, while the figures here investigated are not plane figures.

This object, however, has been accomplished by the most ingenious methods, invented, in great part, by *Monge*, one of the founders of the Polytechnic School at Paris, the first who reduced to a system the elements of this science, called Descriptive Geometry.

Descriptive Geometry is that branch of mathematics which teaches how to represent and determine, by means of drawings on a plane surface, the absolute or relative position of points or magnitudes in space. It is beyond the design of the present work to do more than allude to this interesting and very useful science.

EXERCISES.

615.—1. What is the locus of those points in space, each of which is equally distant from three given points?

2. What is the locus of those points in space, each of which is equally distant from two given planes?

3. What is the locus of those points in space, each of which is equally distant from three given planes?

4. What is the locus of those points in space, each of which is equally distant from two given straight lines which lie in the same plane?

5. What is the locus of those points in space, each of which is equally distant from three given straight lines which lie in the same plane?

6. What is the locus of those points in space, such that the sum of the distances of each from two given planes is equal to a given straight line?

7. If each diedral of a triedral be bisected, the three planes have one common intersection.

8. If a straight line is perpendicular to a plane, every plane parallel to the given line is perpendicular to the given plane.

9. Given any two straight lines in space; either one plane may pass through both, or two parallel planes may pass through them respectively.

10. In the second case of the preceding exercise, a line which is perpendicular to both the given lines is also perpendicular to the two planes.

11. If one face of a triedral is rectangular, then an adjacent diedral angle and its opposite face are either both acute, both right, or both obtuse.

12. Apply to planes, diedrals, and triedrals, respectively, such properties of straight lines, angles, and triangles, as have not already been stated in this chapter, determining, in each case, whether the principle is true when so applied.

CHAPTER X.

POLYEDRONS.

616. A POLYEDRON is a solid, or portion of space, bounded by plane surfaces. Each of these surfaces is a *face*, their several intersections are *edges*, and the points of meeting of the edges are *vertices* of the polyedron.

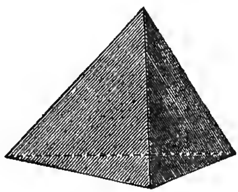
617. Corollary.—The edges being intersections of planes, must be straight lines. It follows that the faces of a polyedron are polygons.

618. A DIAGONAL of a polyedron is a straight line joining two vertices which are not in the same face.

A DIAGONAL PLANE is a plane passing through three vertices which are not in the same face.

TETRAEDRONS.

619. We have seen that three planes can not inclose a space (581). But if any point be taken on each edge of a triedral, a plane passing through these three points would, with the three faces of the triedral, cut off a portion of space, which would be inclosed by four triangular faces.



A TETRAEDRON is a polyedron having four faces.

620. Problem.—*Any four points whatever, which do not all lie in one plane, may be taken as the four vertices of a tetraedron.*

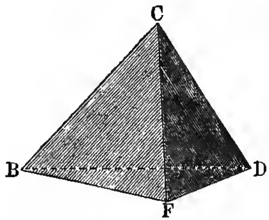
For they may be joined two and two, by straight lines, thus forming the six edges; and these bound the four triangular faces of the figure.

621. Either face of the tetraedron may be taken as the *base*. Then the other faces are called the *sides*, the vertex opposite the base is called the *vertex* of the tetraedron, and the *altitude* is the perpendicular distance from the vertex to the plane of the base. In some cases, the perpendicular falls on the plane of the base produced, as in triangles.

622. Corollary.—If a plane parallel to the base of a tetraedron pass through the vertex, the distance between this plane and the base is the altitude of the tetraedron (574).

623. Theorem.—*There is a point equally distant from the four vertices of any tetraedron.*

In the plane of the face BCF, suppose a circle whose circumference passes through the three points B, C, and F. At the center of this circle, erect a line perpendicular to the plane of BCF.



Every point of this perpendicular is equally distant from the three points B, C, and F (531).

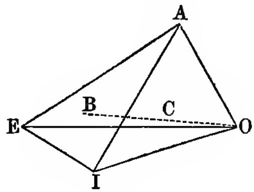
In the same manner, let a line perpendicular to the plane of BDF be erected, so that every point shall be equally distant from the points B, D, and F.

These two perpendiculars both lie in one plane, the plane which bisects the edge BF perpendicularly at its center (520). These two perpendiculars to two oblique planes, being therefore oblique to each other, will meet at some point. This point is equally distant from the four vertices B, C, D, and F.

624. Corollary.—The six planes which bisect perpendicularly the several edges of a tetraedron all meet in one point. But this point is not necessarily within the tetraedron.

625. Theorem.—*There is a point within every tetraedron which is equally distant from the several faces.*

Let AEIO be any tetraedron, and let OB be the straight line formed by the intersection of two planes, one of which bisects the diedral angle whose edge is AO, and the other the diedral whose edge is EO.



Now, every point of the first bisecting plane is equally distant from the faces IAO and EAO (560); and every point of the second bisecting plane is equally distant from the faces EAO and EIO. Therefore, every point of the line BO, which is the intersection of those bisecting planes, is equally distant from those three faces.

Then let a plane bisect the diedral whose edge is EI, and let C be the point where this plane cuts the line BO.

Since every point of this last bisecting plane is equally distant from the faces EAI and EOI, it follows that the point C is equally distant from the four faces of the tetraedron. Since all the bisecting planes are interior, the point found is within the tetraedron.

626. Corollary.—The six planes which bisect the several diedral angles of a tetraedron all meet at one point.

EQUALITY OF TETRAEDRONS.

627. Theorem.—*Two tetraedrons are equal when three faces of the one are respectively equal to three faces of the other, and they are similarly arranged.*

For the three sides of the fourth face, in one, must be equal to the same lines in the other. Hence, the fourth faces are equal. Then each diedral angle in the one is equal to its corresponding diedral angle in the other (599). In a word, every part of the one figure is equal to the corresponding part of the other, and the equal parts are similarly arranged. Therefore, the two tetraedrons are equal.

628. Corollary.—Two tetraedrons are equal when the six edges of the one are respectively equal to those of the other, and they are similarly arranged.

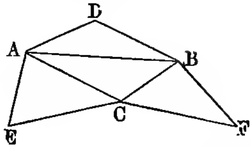
629. Corollary.—Two tetraedrons are equal when two faces and the included diedral of the one are respectively equal to those parts of the other, and they are similarly arranged.

630. Corollary.—Two tetraedrons are equal when one face and the adjacent diedrals of the one are respectively equal to those parts of the other, and they are similarly arranged.

631. When tetraedrons are composed of equal parts in reverse order, they are symmetrical.

MODEL TETRAEDRON.

632. The student may easily construct a model of a tetraedron when the six edges are given. First, with three of the edges which are sides of one face, draw the triangle, as ABC. Then, on each side of this first triangle, as a base, draw a triangle equal to the corresponding face; all of which can be done, for the edges, that is, the sides of these triangles, are given. Then, cut out the whole figure from the paper and carefully fold it at the lines AB, BC, and CA. Since BF is equal to BD, CF to CE, and AD to AE, the points F, D, and E may be united to form a vertex.



In this way models of various forms may be made with more accuracy than in wood, and the student may derive much help from the work.

But he must never forget that the geometrical figure exists only as an intellectual conception. To assist him in this, he should strive to generalize every demonstration, stating the argument without either model or diagram, as in the demonstration last given.

To construct models of symmetrical tetraedrons, the drawings may be equal, but the folding should, in the one case, be up, and in the other, down.

SIMILAR TETRAEDRONS.

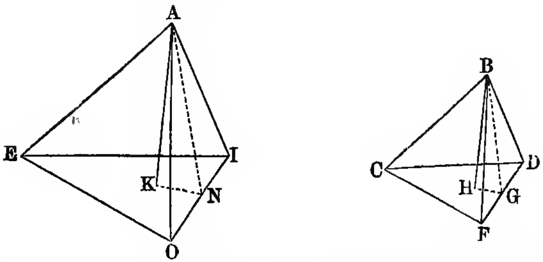
633. Since similarity consists in having the same form, so that every difference of direction in one of two similar figures has its corresponding equal difference of direction in the other, it follows that when two polyedrons are similar, their homologous faces are similar polygons, their homologous edges are of equal diedral angles, and their homologous vertices are of equal polyedrals.

634. Theorem.—*When two tetraedrons are similar, any edge or other line in the one is to the homologous line in the second, as any other line in the first is to its homologous line in the second.*

If the proportion to be proved is between sides of homologous triangles, it follows at once from the similarity of the triangles.

When the edges taken in one of the tetraedrons are not sides of one face; as,

AE : BC :: IO : DF,



then, AE : BC :: IE : CD, as just proved,
and IO : DF :: IE : CD.
Therefore, AE : BC :: IO : DF.

Again, suppose it is to be proved that the altitudes AK and BH have the same ratios as two homologous edges. AK and BH are perpendicular lines let fall from the homologous points A and B on the opposite faces. From K let the perpendicular KN fall upon the edge IO. Join AN, and from H let the perpendicular HG fall upon DF, which is homologous to IO. Join BG.

Now, the planes AKN and EIO are perpendicular to each other (556), and the line IN in one of them is, by construction, perpendicular to their intersection KN. Hence, IN is perpendicular to the plane AKN (557). Therefore, the line AN is perpendicular to IN, and the diedral whose edge is IO is measured by the angle ANK. In the same way, it is proved that the diedral whose edge is DF, is measured by the angle BGH. But these two diedrals, being homologous, are equal, the angles ANK and BGH are equal, and the right angled triangles AKN and BHG are similar. Therefore,

AK : BH :: AN : BG.

Also, the right angled triangles ANI and BGD are similar, since, by hypothesis, the angles AIN and BDG are equal. Hence,

AI : BD :: AN : BG.

Therefore, AK : BH :: AI : BD.

Thus, by the aid of similar triangles, it may be proved that any two homologous lines, in two similar tetraedrons, have the same ratio as two homologous edges.

635. Theorem.—*Two tetraedrons are similar when their faces are respectively similar triangles, and are similarly arranged.*

For we know, from the similarity of the triangles, that every line made on the surface of one may have its homologous line in the second, making angles equal to those made by the first line.

If lines be made through the figure, it may be shown, by the aid of auxiliary lines, as in the corresponding proposition of similar triangles, that every possible angle in the one figure has its homologous equal angle in the other.

The student may draw the diagrams, and go through the details of the demonstration.

636. If the similar faces were not arranged similarly, but in reverse order, the tetraedrons would be *symmetrically similar.*

637. Corollary.—Two tetraedrons are similar when three faces of the one are respectively similar to those of the other, and they are similarly arranged. For the fourth faces, having their sides proportional, are similar also.

638. Corollary.—Two tetraedrons are similar when two triedral vertices of the one are respectively equal to two of the other, and they are similarly arranged.

639. Corollary.—Two tetraedrons are similar when the edges of one are respectively proportional to those of the other, and they are similarly arranged.

640. Theorem.—*The areas of homologous faces of similar tetraedrons are to each other as the squares of their edges.*

This is only a corollary of the theorem that the areas of similar triangles are to each other as the squares of their sides.

641. Corollary.—The areas of homologous faces of similar tetraedrons are to each other as the squares of any homologous lines.

642. Corollary.—The area of any face of one tetraedron is to the area of a homologous face of a similar tetraedron, as the area of any other face of the first is to the area of the homologous face of the second.

643. Corollary.—The area of the entire surface of one tetraedron is to that of a similar tetraedron as the squares of homologous lines.

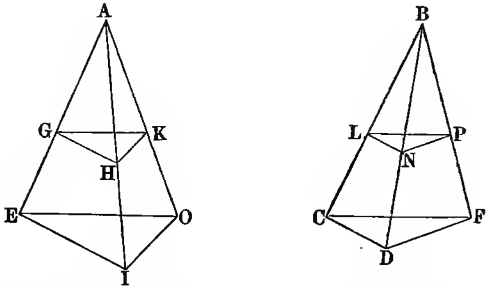
TETRAEDRONS CUT BY A PLANE.

644. Theorem.—*If a plane cut a tetraedron parallel to the base, the tetraedron cut off is similar to the whole.*

For each triangular side is cut by a line parallel to its base (572), thus making all the edges of the two tetraedrons respectively proportional.

645. Theorem.—*If two tetraedrons, having the same altitude and their bases on the same plane, are cut by a plane parallel to their bases, the areas of the sections will have the same ratio as the areas of the bases.*

If a plane parallel to the bases pass through the vertex A, it will also pass through the vertex B (622). But such a plane is parallel to the cutting plane GHP (566).



Therefore, the tetraedrons AGHK and BLNP have equal altitudes.

The tetraedrons AEIO and AGHK are similar (644). Therefore, EIO, the base of the first, is to GHK, the base of the second, as the square of the altitude of the first is to the square of the altitude of the second (641). For a like reason, the base CDF is to the base LNP as the square of the greater altitude is to the square of the less.

Therefore, EIO : GHK : : CDF : LNP.

By alternation,

EIO : CDF : : GHK : LNP.

646. Corollary.—When the bases are equivalent the sections are equivalent.

647. Corollary.—When the bases are equal the sections are equal. For they are similar and equivalent.

REGULAR TETRAEDRON.

648. There is one form of the tetraedron which deserves particular notice. It has all its faces equilateral. This is called a regular tetraedron.

649. Corollary.—It follows, from the definition, that the faces are equal triangles, the vertices are of equal triedrals, and the edges are of equal diedral angles.

650. The area of the surface of a tetraedron is found by taking the sum of the areas of the four faces. When two or more of them are equal, the process is shortened by multiplication. But the discussion of this matter will be included in the subject of the areas of pyramids.

The investigation of the measures of volumes will be given in another connection.

EXERCISES.

651.—1. State other cases, when two tetraedrons are similar, in addition to those given, Articles 635 to 639.

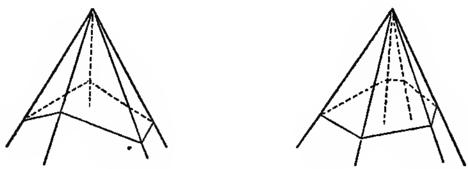
2. In any tetraedron, the lines which join the centers of the opposite edges bisect each other.

3. If one of the vertices of a tetraedron is a trirectangular triedral, the square of the area of the opposite face is equal to the sum of the squares of the areas of the other three faces.

PYRAMIDS.

652. If a polyedral is cut by a plane which cuts its several edges, the section is a polygon, and a portion of space is cut off, which is called a pyramid.

A Pyramid is a polyedron having for one face any



polygon, and for its other faces, triangles whose vertices meet at one point.

The polygon is the *base* of the pyramid, the triangles are its *sides*, and their intersections are the *lateral edges* of the pyramid. The vertex of the polyedral is the *vertex* of the pyramid, and the perpendicular distance from that point to the plane of the base is its *altitude*.

Pyramids are called triangular, quadrangular, pentagonal, etc., according to the polygon which forms the base. The tetraedron is a triangular pyramid.

653. Problem.—*Every pyramid can be divided into the same number of tetraedrons as its base can be into triangles.*

Let a diagonal plane pass through the vertex of the pyramid and each diagonal of the base, and the solution is evident.

EQUAL PYRAMIDS.

654. Theorem.—*Two pyramids are equal when the base and two adjacent sides of the one are respectively equal to the corresponding parts of the other, and they are similarly arranged.*

For the triedrals formed by the given faces in the two must be equal, and may therefore coincide; and the given faces will also coincide, being equal. But now the vertices and bases of the two pyramids coincide. These include the extremities of every edge. Therefore, the edges coincide; also the faces, and the figures throughout.

SIMILAR PYRAMIDS.

655. Theorem.—*Two similar pyramids are composed of tetraedrons respectively similar, and similarly arranged; and, conversely, two pyramids are similar when composed of similar tetraedrons, similarly arranged.*

656. Theorem.—*When a pyramid is cut by a plane parallel to the base, the pyramid cut off is similar to the whole.*

These theorems may be demonstrated by the student. Their demonstration is like that of analogous propositions in triangles and tetraedrons.

REGULAR PYRAMIDS.

657. A Regular Pyramid is one whose base is a regular polygon, and whose vertex is in the line perpendicular to the base at its center.

658. Corollary.—The lateral edges of a regular pyramid are all equal (529), and the sides are equal isosceles triangles.

659. The Slant Hight of a regular pyramid is the perpendicular let fall from the vertex upon one side of the base. It is therefore the altitude of one of the sides of the pyramid.

660. Theorem.—*The area of the lateral surface of a regular pyramid is equal to half the product of the perimeter of the base by the slant hight.*

The area of each side is equal to half the product of its base by its altitude (386). But the altitude of each of the sides is the slant hight of the pyramid, and the sum of all the bases of the sides is the perimeter of the base of the pyramid.

Therefore, the area of the lateral surface of the pyramid, which is the sum of all the sides, is equal to half the product of the perimeter of the base by the slant hight.

661. When a pyramid is cut by a plane parallel to the base, that part of the figure between this plane and the base is called a *frustum* of a pyramid, or a *truncated* pyramid.

662. Corollary.—The sides of a frustum of a pyramid are trapezoids (572); and the sides of the frustum of a regular pyramid are equal trapezoids.

663. The section made by the cutting plane is called the *upper base* of the frustum. The *slant hight* of the frustum of a regular pyramid is that part of the slant hight of the original pyramid which lies between the bases of the frustum. It is therefore the altitude of one of the lateral sides.

664. Theorem.—*The area of the lateral surface of the frustum of a regular pyramid is equal to half the product of the sum of the perimeters of the bases by the slant hight.*

The area of each trapezoidal side is equal to half the product of the sum of its parallel bases by its altitude (392), which is the slant hight of the frustum. Therefore, the area of the lateral surface, which is the sum of all these equal trapezoids, is equal to the product of half the sum of the perimeters of the bases of the frustum, multiplied by the slant hight.

665. Corollary.—The area of the lateral surface of a frustum of a regular pyramid is equal to the product of the perimeter of a section midway between the two bases, multiplied by the slant hight. For the perimeter of a section, midway between the two bases, is equal to half the sum of the perimeters of the bases.

666. Corollary.—The area of the lateral surface of a regular pyramid is equal to the product of the slant hight by the perimeter of a section, midway between the vertex and the base. For the perimeter of the middle section is one-half the perimeter of the base.

MODEL PYRAMIDS.

667. The student may construct a model of a regular pyramid. First, draw a regular polygon of any number of sides. Upon these sides, as bases, draw equal isosceles triangles, taking care that their altitude be greater than the apothem of the base. The figure may then be cut out and folded.

EXERCISES.

668.—1. Find the area of the surface of a regular octagonal pyramid whose slant hight is 5 inches, and a side of whose base is 2 inches.

2. What is the area in square inches of the entire surface of a regular tetraedron, the edge being one inch? *Ans.* $\sqrt{3}$.

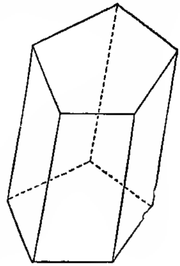
3. A pyramid is regular when its sides are equal isosceles triangles, whose bases form the perimeter of the base of the pyramid.

4. State other cases of equal pyramids, in addition to those given, Article 654.

5. When two pyramids of equal altitude have their bases in the same plane, and are cut by a plane parallel to their bases, the areas of the sections are proportional to the areas of the bases.

PRISMS.

669. A Prism is a polyedron which has two of its faces equal polygons lying in parallel planes, and the other faces parallelograms. Its possibility is shown by supposing two equal and parallel polygons lying in two parallel planes (569). The equal sides being parallel, let planes unite them. The figure thus formed on each plane is a parallelogram, for it has two opposite sides equal and parallel.



The parallel polygons are called the *bases*, the parallelograms the *sides* of the prism, and the intersections of the sides are its *lateral edges.*

The *altitude* of a prism is the perpendicular distance between the planes of its bases.

670. Corollary.—The lateral edges of a prism are all parallel to each other, and therefore equal to each other (573).

671. A RIGHT PRISM is one whose lateral edges are perpendicular to the bases.

A REGULAR PRISM is a right prism whose base is a regular polygon.

672. Corollary.—The altitude of a right prism is equal to one of its lateral edges; and the sides of a right prism are rectangles. The sides of a regular prism are equal.

673. Theorem.—*If two parallel planes pass through a prism, so that each plane cuts every lateral edge, the sections made by the two planes are equal polygons.*

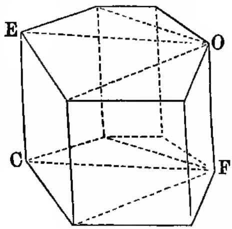
Each side of one of the sections is parallel to the corresponding side of the other section, since they are the intersections of two parallel planes by a third. Hence, that portion of each side of the prism which is between the secant planes, is a parallelogram. Since the sections have their sides respectively equal and parallel, their angles are respectively equal. Therefore, the polygons are equal.

674. Corollary.—The section of a prism made by a plane parallel to the base is equal to the base, and the given prism is divided into two prisms. If two parallel planes cut a prism, as stated in the above theorem, that part of the solid between the two secant planes is also a prism.

HOW DIVISIBLE.

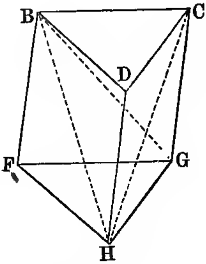
675. Problem.—*Every prism can be divided into the same number of triangular prisms as its base can be into triangles.*

If homologous diagonals be made in the two bases, as EO and CF, they will lie in one plane. For CE and OF being parallel to each other (670), lie in one plane. Therefore, through each pair of these homologous diagonals a plane may pass, and these diagonal planes divide the prisms into triangular prisms.



676. Problem.—*A triangular prism may be divided into three tetraedrons, which, taken two and two, have equal bases and equal altitudes.*

Let a diagonal plane pass through the points B, C, and H, making the intersections BH and CH, in the sides DF and DG. This plane cuts off the tetraedron BCDH, which has for one of its faces the base BCD of the prism; for a second face, the triangle BCH, being the section made by the diagonal plane; and for its other two faces, the triangles BDH and CDH, each being half of one of the sides of the prism.



The remainder of the prism is a quadrangular pyramid, having the parallelogram BCGF for its base, and H for its vertex. Let it be cut by a diagonal plane through the points H, G, and B.

This plane separates two tetraedrons, HBCG and HBFG. The two faces, HBC and HBG, of the tetraedron HBCG, are sections made by the diagonal planes; and the two faces, HCG and BCG, are each half of one side of the prism. The tetraedron HBFG has for one of its faces the base HFG of the prism; for a second face, the triangle HBG, being the section made by the diagonal plane; and, for the other two, the triangles HBF and GBF, each being half of one of the sides of the prism.

Now, consider these two tetraedrons as having their bases BCG and BFG. These are equal triangles lying in one plane. The point H is the common vertex, and therefore they have the same altitude; that is, a perpendicular from H to the plane BCGF.

Next, consider the first and last tetraedrons described, HBCD and BFGH, the former as having BCD for its base, and H for its vertex; the latter as having FGH for its base, and B for its vertex. These bases are equal, being the bases of the given prism. The vertex of each is in the plane of the base of the other. Therefore, the altitudes are equal, being the distance between these two planes.

Lastly, consider the tetraedrons BCDH and BCGH as having their bases CDH and CGH. These are equal triangles lying in one plane. The tetraedrons have the common vertex B, and hence have the same altitude.

677. Corollary.—Any prism may be divided into tetraedrons in several ways; but the methods above explained are the simplest.

678. Remark.—On account of the importance of the above problem in future demonstrations, the student is advised to make a model triangular prism, and divide it into tetraedrons. A potato may be used for this purpose. The student will derive most benefit from those models and diagrams which he makes himself.

EQUAL PRISMS.

679. Theorem.—*Two prisms are equal, when a base and two adjacent sides of the one are respectively equal to the corresponding parts of the other, and they are similarly arranged.*

For the triedrals formed by the given faces in the two prisms must be equal (599), and may therefore be made to coincide. Then the given faces will also coincide, being equal. These coincident points include all of one base, and several points in the second. But the second bases have their sides respectively equal, and parallel to those of the first. Therefore, they also coincide, and the two prisms having both bases coincident, must coincide throughout.

680. Corollary.—Two right prisms are equal when they have equal bases and the same altitude.

681. The theory of similar prisms presents nothing difficult or peculiar. The same is true of symmetrical prisms, and of symmetrically similar prisms.

AREA OF THE SURFACE.

682. Theorem.—*The area of the lateral surface of a prism is equal to the product of one of the lateral edges by the perimeter of a section, made by a plane perpendicular to those edges.*

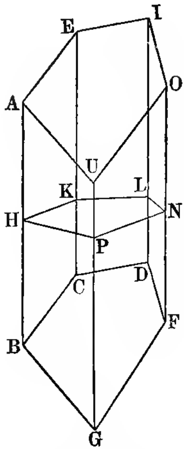
Since the lateral edges are parallel, the plane HN, perpendicular to one, is perpendicular to all of them. Therefore, the sides of the polygon, HK, KL, etc., are severally perpendicular to the edges of the prism which they unite (519).

Then, in order to measure the area of each face of the prism, we take one edge of the prism as the base of the parallelogram, and one side of the polygon HN as its altitude.

Thus,

area $AG = AB \times HP$,

area $EB = EC \times HK$, etc.



By addition, the sum of the areas of these parallelograms is the lateral surface of the prism, and the sum of the altitudes of the parallelograms is the perimeter of the polygon HN. Then, since the edges are equal, the area of all the sides is equal to the product of one edge, multiplied by the perimeter of the polygon.

683. Corollary.—The area of the lateral surface of a right prism is equal to the product of the altitude by the perimeter of the base.

684. Corollary.—The area of the entire surface of a regular prism is equal to the product of the perimeter of the base by the sum of the altitude of the prism and the apothem of the base.

EXERCISES.

685.—1. A right prism has less surface than any other prism of equal base and equal altitude; and a regular prism has less surface than any other right prism of equivalent base and equal altitude.

2. A regular pyramid and a regular prism have equal hexagonal bases, and altitudes equal to three times the radius of the base; required the ratio of the areas of their lateral surfaces.

3. Demonstrate the principle stated in Article 683, without the aid of Article 682.

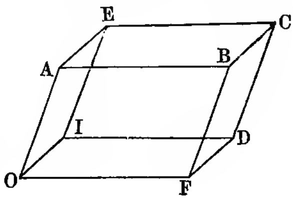
MEASURE OF VOLUME.

686. A PARALLELOPIPED is a prism whose bases are parallelograms. Hence, a parallelopiped is a solid inclosed by six parallelograms.

687. Theorem.—*The opposite sides of a parallelopiped are equal.*

For example, the faces AI and BD are equal.

For IO and DF are equal, being opposite sides of the parallelogram IF. For a like reason, EI is equal to CD. But, since these equal sides are also parallel, the included angles EIO and CDF are equal. Hence, the parallelograms are equal.



688. Corollary.—Any two opposite faces of a parallelopiped may be assumed as the bases of the figure.

689. A parallelopiped is called *right* in the same case as any other prism. When the bases also are rectangles, it is called *rectangular.* Then all the faces are rectangles.

690. A CUBE is a rectangular parallelopiped whose length, breadth, and altitude are equal. Then a cube is a solid, bounded by six equal squares. All its vertices, being trirectangular triedrals, are equal (602). All its edges are of right diedral angles, and therefore equal (555).

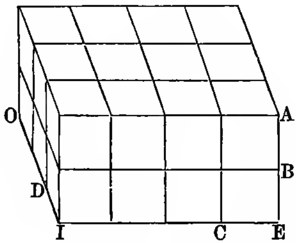
The cube has the simplest form of all geometrical solids. It holds the same rank among them that the square does among plane figures, and the straight line among lines.

The cube is taken, therefore, as the unit of measure of volume. That is, whatever straight line is taken as the unit of length, the cube whose edge is of that length is the unit of volume, as the square whose side is of that length is the measure of area.

VOLUME OF PARALLELOPIPEDS.

691. Theorem.—*The volume of a rectangular parallelopiped is equal to the product of its length, breadth, and altitude.*

In the measure of the rectangle, the product of one line by another was explained. Here we have three lines used with a similar meaning. That is, the number of cubical units contained in a rectangular parallelopiped is equal to the product of the numbers of linear units in the length, the breadth, and the altitude.



If the altitude AE, the length EI, and the breadth IO, have a common measure, let each be divided by it; and let planes, parallel to the faces of the prism, pass through all the points of division, B, C, D, etc.

By this construction, all the angles formed by these planes and their intersections are right angles, and each of the intercepted lines is equal to the linear unit used in dividing the edges of the prism. Therefore, the prism is divided into equal cubes. The number of these at the base is equal to the number of rows, multiplied by the number in each row; that is, the product of the length by the breadth. There are as many layers of cubes as there are linear units of altitude. Therefore, the whole number is equal to the product of the length, breadth, and altitude. In the diagram, the dimensions being four, three, and two, the volume is twenty-four.

But if the length, breadth, and altitude have no common measure, a linear unit may be taken, successively smaller and smaller. In this, we would not take the whole of the linear dimensions, nor would we measure the whole of the prism. But the remainder of both would grow less and less. The part of the prism measured at each step, would be measured exactly by the principle just demonstrated.

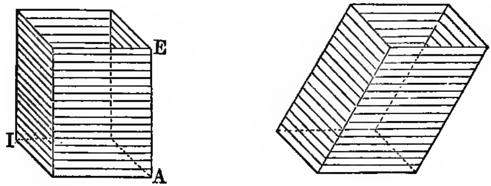
By these successive diminutions of the unit, we can make the part measured approach to the whole prism as nearly as we please. In a word, the whole is the limit of the parts measured; and since the principle demonstrated is true up to the limit, it must be true at the limit. Therefore, the rectangular parallelopiped is measured by the product of its length, breadth, and altitude.

692. Theorem.—*The volume of any parallelopiped is equal to the product of its length, breadth, and altitude.*

Inasmuch as this has just been demonstrated for the rectangular parallelopiped, it will be sufficient to show that any parallelopiped is equivalent to a rectangular one having the same linear dimensions.

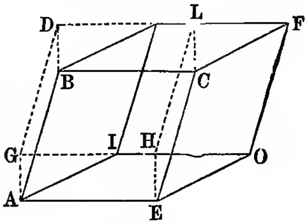
Suppose the lower bases of the two prisms to be placed on the same plane. Then their upper bases must also be in one plane, since they have the same altitude. Let the altitude AE be divided into an infinite number of equal parts, and through each point of division pass a plane parallel to the base AI.

Now, every section in either prism is equal to the base; but the bases of the two prisms, having the same length and breadth, are equivalent. The several partial infinitesimal prisms are reduced to equivalent fig-



ures. Although they are not, strictly speaking, parallelograms, yet their altitudes being infinitesimal, there can be no error in considering them as plane figures; which, being equal to their respective bases, are equivalent. Then, the number of these is the same in each prism. Therefore, the sum of the whole, in one, is equivalent to the sum of the whole, in the other; that is, the two parallelopipeds are equivalent.

Besides the above demonstration by the method of infinites, the theorem may be demonstrated by the ordinary method of reasoning, which is deduced from principles that depend upon the superposition and coincidence of equal figures, as follows:



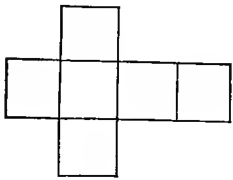
Let AF be any oblique parallelopiped. It may be shown to be equivalent to the parallelopiped AL, which has a rectangular base, AH, since the prism LHEO is equal to the prism DGAI. But the parallelopipeds AF and AL have the same length, breadth, and altitude.

By similar reasoning, the prism AL may be shown to be equivalent to a prism of the same base and altitude, but with two of its opposite sides rectangular. This third prism may then be shown to be equivalent to a fourth, which is rectangular, and has the same dimensions as the others.

693. Corollary.—The volume of a cube is equal to the third power of its edge. Thence comes the name of cube, to designate the third power of a number.

MODEL CUBES.

694. Draw six equal squares, as in the diagram. Cut out the figure, fold at the dividing lines, and glue the edges. It is well to have at least eight of one size.



695. Corollary.—The volume of any parallelopiped is equal to the product of its base by its altitude.

696. Corollary.—The volumes of any two parallelopipeds are to each other as the products of their three dimensions.

VOLUME OF PRISMS.

697. Theorem.—*The volume of any triangular prism is equal to the product of its base by its altitude.*

The base of any right triangular prism may be considered as one-half of the base of a right parallelopiped. Then the whole parallelopiped is double the given prism, for it is composed of two right prisms having equal bases and the same altitude, of which the given prism is one. Therefore, the given prism is measured by half the product of its altitude by the base of the parallelopiped; that is, by the product of its own base and altitude.

If the given prism be oblique, it may be shown, by demonstrations similar to the first of those in Article 692, to be equivalent to a right prism having the same base and altitude.

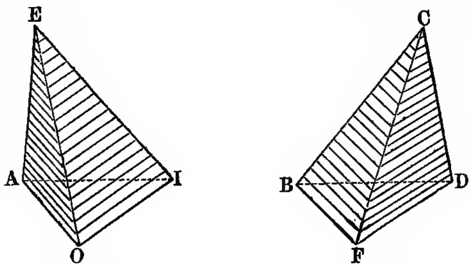
698. Corollary.—The volume of any prism is equal to the product of its base by its altitude. For any prism is composed of triangular prisms, having the common altitude of the given prism, and the sum of their bases forming the given base.

699. Corollary.—The volume of a triangular prism is equal to the product of one of its lateral edges multiplied by the area of a section perpendicular to that edge.

VOLUME OF TETRAEDRONS.

700. Theorem.—*Two tetraedrons of equivalent bases and of the same altitude are equivalent.*

Suppose the bases of the two tetraedrons to be in the



same plane. Then their vertices lie in a plane parallel to the bases, since the altitudes are equal. Let the edge AE be divided into an infinite number of parts, and through each point of division pass a plane parallel to the base AIO.

Now, the several infinitesimal frustums into which the two figures are divided may, without error, be considered as plane figures, since their altitudes are infinitesimal. But each section of one tetraedron is equivalent to the section made by the same plane in the other tetraedron. Therefore, the sum of all the infinitesimal frustums in the one figure is equivalent to the sum of all in the other; that is, the two tetraedrons are equivalent.

701. Theorem.—*The volume of a tetraedron is equal to one-third of the product of the base by the altitude.*

Upon the base of any given tetraedron, a triangular prism may be erected, which shall have the same altitude, and one edge coincident with an edge of the tetraedron. This prism may be divided into three tetraedrons, the given one and two others, which, taken two and two, have equal bases and altitudes (676).

Then, these three tetraedrons are equivalent (700); and the volume of the given tetraedron is one-third of the volume of the prism; that is, one-third of the product of its base by its altitude.

VOLUME OF PYRAMIDS.

702. Corollary.—The volume of any pyramid is equal to one-third of the product of its base by its altitude. For any pyramid is composed of triangular pyramids; that is, of tetraedrons having the common altitude of the given pyramid, and the sum of their bases forming the given base (653).

703. Corollary.—The volumes of two prisms of equivalent bases are to each other as their altitudes, and the volumes of two prisms of equal altitudes are to each other as their bases. The same is true of pyramids.

704. Corollary.—Symmetrical prisms are equivalent. The same is true of symmetrical pyramids.

705. The volume of a frustum of a pyramid is found by subtracting the volume of the pyramid cut off from the volume of the whole. When the altitude of the whole is not given, it may be found by this proportion: the area of the lower base of the frustum is to the area of its upper base, which is the base of the part cut off, as the square of the whole altitude is to the square of the altitude of the part cut off.

EXERCISES.

706.—1. What is the ratio of the volumes of a pyramid and prism having the same base and altitude?

2. If two tetraedrons have a triedral vertex in each equal, their volumes are in the ratio of the products of the edges which contain the equal vertices.

3. The plane which bisects a diedral angle of a tetraedron, divides the opposite edge in the ratio of the areas of the adjacent faces.

SIMILAR POLYEDRONS.

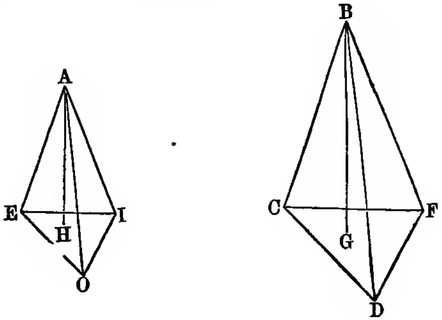
707. The propositions (640 to 643) upon the ratios of the areas of the surfaces of similar tetraedrons, may be applied by the student to any similar polyedrons. These propositions and the following are equally applicable to polyedrons that are symmetrically similar.

708. Problem.—*Any two similar polyedrons may be divided into the same number of similar tetraedrons, which shall be respectively similar, and similarly arranged.*

For, after dividing one into tetraedrons, the construc tion of the homologous lines in the other will divide it in the same manner. Then the similarity of the respective tetraedrons follows from the proportionality of the lines.

709. Theorem.—*The volumes of similar polyedrons are proportional to the cubes of homologous lines.*

First, suppose the figures to be tetraedrons. Let AH and BG be the altitudes.



Then (641), EIO : CDF : : EI^2 : CF^2 : : AH^2 : BG^2

By the proportionality of homologous lines, (634),

$\frac{1}{3}$ AH : $\frac{1}{3}$ BG : : EI : CF : : AH : BG.

Multiplying these proportions (701), we have

AEIO : BCFD : : EI^3 : CF^3 : : AH^3 : BG^3,

or, as the cubes of any other homologous lines.

Next, let any two similar polyedrons be divided into the same number of tetraedrons. Then, as just proved, the volumes of the homologous parts are proportional to the cubes of the homologous lines. By arranging these in a continued proportion, as in Article 436, we may show that the volume of either polyedron is to the volume of the other as the cube of any line of the first is to the cube of the homologous line of the second.

710. Notice that in the measure of every area there are two linear dimensions; and in the measure of every volume, three linear, or one linear and one superficial.

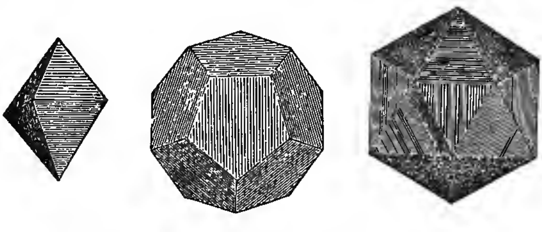
EXERCISE.

711. What is the ratio between the edges of two cubes, one of which has twice the volume of the other?

This problem of the duplication of the cube was one of the celebrated problems of ancient times. It is said that the oracle of Apollo at Delphos, demanded of the Athenians a new altar, of the same shape, but of twice the volume of the old one. The efforts of the Greek geometers were chiefly aimed at a graphic solution; that is, the edge of one cube being given, to draw a line equal to the edge of the other, using no instruments but the rule and compasses. In this they failed. The student will find no difficulty in making an arithmetical solution, within any desired degree of approximation.

REGULAR POLYEDRONS.

712. A Regular Polyedron is one whose faces are equal and regular polygons, and whose vertices are equal polyedrals.



The regular tetraedron and the cube, or regular hexaedron, have been described.

The regular *octaedron* has eight, the *dodecaedron* twelve, and the *icosaedron* twenty faces.

The class of figures here defined must not be confounded with regular pyramids or prisms.

713. Problem.—*It is not possible to make more than five regular polyedrons.*

First, consider those whose faces are triangles. Each angle of a regular triangle is one-third of two right angles. Either three, four, or five of these may be joined to form one polyedral vertex, the sum being, in each case, less than four right angles (612). But the sum of six such angles is not less than four right angles. Therefore, there can not be more than three kinds of regular polyedrons whose faces are triangles, viz.: the tetraedron, where three plane angles form a vertex; the octaedron, where four, and the icosaedron, where five angles form a vertex.

The same kind of reasoning shows that only one regular polyedron is possible with square faces, the cubé; and only one with pentagonal faces, the dodecaedron.

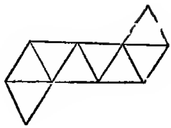
Regular hexagons can not form the faces of a regular polyedron, for three of the angles of a regular hexagon are together not less than four right angles; and therefore they can not form a vertex.

So much the more, if the polygon has a greater number of sides, it will be impossible for its angles to be the faces of a polyedral. Therefore, no polyedron is possible, except the five that have been described.

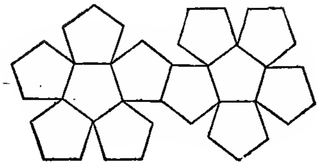
MODEL REGULAR POLYEDRONS.

714. The possibility of regular polyedrons of eight, of twelve, and of twenty sides is here assumed, as the demonstration would occupy more space than the principle is worth. However, the student may construct models of these as follows. Plans for the regular tetraedron and the cube have already been given.

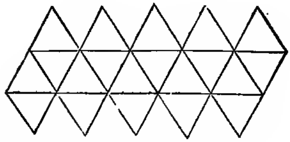
For the octaedron, draw eight equal regular triangles, as in the diagram.



For the dodecaedron, draw twelve equal regular pentagons, as in the diagram.



For the icosaedron, draw twenty equal regular triangles, as in the diagram.



There are many crystals, which, though not regular, in the geometrical rigor of the word, yet present a certain regularity of shape.

EXERCISES.

715.—1. How many edges and how many vertices has each of the regular polyedrons?

2. Calling that point the *center of a triangle* which is the intersection of straight lines from each vertex to the center of the opposite side; then, demonstrate that the four lines which join the vertices of a tetraedron to the centers of the opposite faces, inter sect each other in one point.

3. In what ratio do the lines just described in the tetraedron divide each other?

4. The opposite vertices of a parallelopiped are symmetrical triedrals.

5 The diagonals of a parallelopiped bisect each other; the lines which join the centers of the opposite edges bisect each other; the lines which join the centers of the opposite faces bi-

sect each other; and the point of intersection is the same for all these lines.

6. The diagonals of a rectangular parallelopiped are equal.

7. The square of the diagonal of a rectangular parallelopiped is equivalent to the sum of the squares of its length, breadth, and altitude.

8. A cube is the largest parallelopiped of the same extent of surface.

9. If a right prism is symmetrical to another, they are equal.

10. Within any regular polyedron there is a point equally distant from all the faces, and also from all the vertices.

11. Two regular polyedrons of the same number of faces are similar.

12. Any regular polyedron may be divided into as many regular and equal pyramids as it has faces.

13. Two different tetraedrons, and only two, may be formed with the same four triangular faces; and these two tetraedrons are symmetrical.

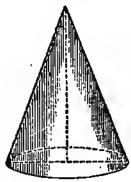
14. The area of the lower base of a frustum of a pyramid is five square feet, of the upper base one and four-fifths square feet, and the altitude is two feet; required the volume.

CHAPTER XI.

SOLIDS OF REVOLUTION.

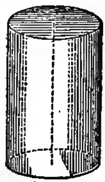
716. Of the infinite variety of forms there remain but three to be considered in this elementary work. These are formed or generated by the revolution of a plane figure about one of its lines as an axis. Figures formed in this way are called *solids of revolution.*

717. A Cone is a solid formed by the revolution of a right angled triangle about one of its legs as an axis. The other leg revolving describes a plane surface (521). This surface is also a circle, having for its radius the leg by which it is described. The hypotenuse describes a curved surface.



The plane surface of a cone is called its *base.* The opposite extremity of the axis is the *vertex.* The *altitude* is the distance from the vertex to the base, and the *slant hight* is the distance from the vertex to the circumference of the base.

718. A Cylinder is a solid described by the revolution of a rectangle about one of its sides as an axis. As in the cone, the sides adjacent to the axis describe circles, while the opposite side describes a curved surface.



The plane surfaces of a cylinder are called its *bases,* and the perpendicular distance between them is its *altitude.*

These figures are strictly a regular cone and a regular cylinder, yet but one word is used to denote the figures defined, since other cones and cylinders are not usually discussed in Elementary Geometry. The sphere, which is described by the revolution of a semicircle about the diameter, will be considered separately.

719. As the curved surfaces of the cone and of the cylinder are generated by the motion of a straight line, it follows that each of these surfaces is straight in one direction.

A straight line from the vertex of the cone to the circumference of the base, must lie wholly in the surface. So a straight line, perpendicular to the base of a cylinder at its circumference, must lie wholly in the surface. For, in each case, these positions had been occupied by the generating lines.

One surface is *tangent* to another when it meets, but being produced does not cut it. The place of contact of a plane with a conical or cylindrical surface, must be a straight line; since, from any point of one of those surfaces, it is straight in one direction.

CONIC SECTIONS.

720. Every point of the line which describes the curved surface of a cone, or of a cylinder, moves in a plane parallel to the base (565). Therefore, if a cone or a cylinder be cut by a plane parallel to the base, the section is a circle.

If we conceive a cone to be cut by a plane, the curve formed by the intersection will be different according to the position of the cutting plane. There are three different modes in which it is possible for the intersection to take place. The curves thus formed are the ellipse, parabola, and hyperbola.

These *Conic Sections* are not usually considered in Elementary Geometry, as their properties can be better investigated by the application of algebra.

CONES.

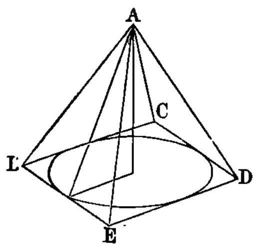
721. A cone is said to be *inscribed* in a pyramid, when their bases lie in one plane, and the sides of the pyramid are tangent to the curved surface of the cone. The pyramid is said to be *circumscribed* about the cone.

A cone is said to be *circumscribed* about a pyramid, when their bases lie in one plane, and the lateral edges of the pyramid lie in the curved surface of the cone. Then the pyramid is *inscribed* in the cone.

722. Theorem.—*A cone is the limit of the pyramids which can be circumscribed about it; also of the pyramids which can be inscribed in it.*

Let ABCDE be any pyramid circumscribed about a cone.

The base of the cone is a circle inscribed in the base of the pyramid. The sides of the pyramid are tangent to the surface of the cone.



Now, about the base of the cone there may be described a polygon of double the number of sides of the first, each alternate side of the second polygon coinciding with a side of the first. This second polygon may be the base of a pyramid, having its vertex at A. Since the sides of its bases are tangent to the base of the cone, every side of the pyramid is tangent to the curved surface of the cone. Thus the second pyramid is circumscribed about the cone, but is itself within the first pyramid.

By increasing the number of sides of the pyramid, it can be made to approximate to the cone within less than any appreciable difference. Then, as the base of the cone is the limit of the bases of the pyramids, the cone itself is also the limit of the pyramids.

Again, let a polygon be inscribed in the base of the cone. Then, straight lines joining its vertices with the vertex of the cone form the lateral edges of an inscribed pyramid. The number of sides of the base of the pyramid, and of the pyramid also, may be increased at will. It is evident, therefore, that the cone is the limit of pyramids, either circumscribed or inscribed.

723. Corollary.—The area of the curved surface of a cone is equal to one-half the product of the slant hight by the circumference of the base (660). Also, it is equal to the product of the slant hight by the circumference of a section midway between the vertex and the base (666).

724. Corollary.—The area of the entire surface of a cone is equal to half of the product of the circumference of the base by the sum of the slant hight and the radius of the base (499).

725. Corollary.—The volume of a cone is equal to one-third of the product of the base by the altitude.

726. The frustum of a cone is defined in the same way as the frustum of a pyramid.

727. Corollary.—The area of the curved surface of the frustum of a cone is equal to half the product of its slant hight by the sum of the circumferences of its bases (664). Also, it is equal to the product of its slant hight by the circumference of a section midway between the two bases (665).

728. Corollary.—If a cone be cut by a plane parallel to the base, the cone cut off is similar to the whole (656).

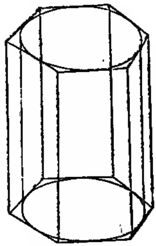
EXERCISES.

729.—1. Two cones are similar when they are generated by similar triangles, homologous sides being used for the axes.

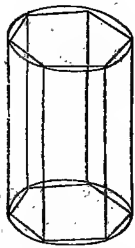
2. A section of a cone by a plane passing through the vertex, is an isosceles triangle.

CYLINDERS.

730. A cylinder is said to be *inscribed* in a prism, when their bases lie in the same planes, and the sides of the prism are tangent to the curved surface of the cylinder. The prism is then said to be *circumscribed* about the cylinder.



A cylinder is said to be *circumscribed* about a prism, when their bases lie in the same planes, and the lateral edges of the prism lie in the curved surface of the cylinder; and the prism is then said to be *inscribed* in the cylinder.



731. Theorem.—*A cylinder is the limit of the prisms which can be circumscribed about it; also of those which can be inscribed in it.*

The demonstration of this theorem is so similar to that of the last, that it need not be repeated.

732. Corollary.—The area of the curved surface of a cylinder is equal to the product of the altitude by the circumference of the base (683).

733. Corollary.—The area of the entire surface of a cylinder is equal to the product of the circumference of the base by the sum of the altitude and the radius of the base (684).

734. Corollary.—The volume of a cylinder is equal to the product of the base by the altitude (698).

MODEL CONES AND CYLINDERS.

735. Models of cones and cylinders may be made from paper, taking a sector of a circle for the curved surface of a cone, and a rectangle for the curved surface of a cylinder. Make the bases separately.

EXERCISES.

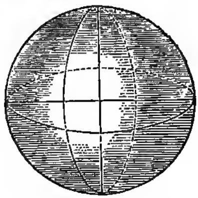
736.—1. Apply to cones and cylinders the principles demonstrated of similar polyedrons.

2. A section of a cylinder made by a plane perpendicular to the base is a rectangle.

3. The axis of a cone or of a cylinder is equal to its altitude.

SPHERES.

737. A Sphere is a solid described by the revolution of a semicircle about its diameter as an axis.



The *center*, *radius*, and *diameter* of the sphere are the same as those of the generating circle. The spherical surface is described by the circumference.

738. Corollary.—Every point on the surface of the sphere is equally distant from the center.

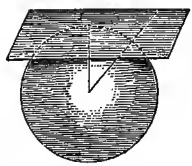
This property of the sphere is frequently given as its definition.

739. Corollary.—All radii of the same sphere are equal. The same is true of the diameters.

740. Corollary.—Spheres having equal radii are equal.

741. Corollary.—A plane passing through the center of a sphere divides it into equal parts. The halves of a sphere are called *hemispheres.*

742. Theorem.—*A plane which is perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*



For if straight lines extend from the center of the sphere to any other point of the plane, they are oblique and longer than the radius, which is perpendicular (530). Therefore, every point of the plane except one is beyond the surface of the sphere, and the plane is tangent.

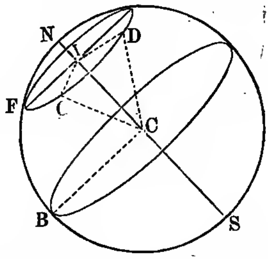
743. Corollary.—The spherical surface is curved in every direction. Unlike those surfaces which are generated by the motion of a straight line, every possible section of it is a curve.

SECANT PLANES.

744. Theorem.—*Every section of a sphere made by a plane is a circle.*

If the plane pass through the center of the sphere, every point in the perimeter of the section is equally distant from the center, and therefore the section is a circle.

But if the section do not pass through the center, as DGF, then from the center C let CI fall perpendicularly on the cutting plane. Let radii of the sphere, as CD and CG, extend to different points of the boundary of the section, and join ID and IG.



Now the oblique lines CD and CG being equal, the points D and G must be equally distant from I, the foot of the perpendicular (529). The same is true of all the points of the perimeter DGF. Therefore, DGF is the circumference of a circle of which I is the center.

745. Corollary.—The circle formed by the section through the center is larger than one formed by any plane not through the center. For the radius BC is equal to GC, and longer than GI (104).

746. When the plane passes through the center of a sphere, the section is called a *great circle;* otherwise it is called a *small circle.*

747. Corollary.—All great circles of the same sphere are equal.

748. Corollary.—Two great circles bisect each other, and their intersection is a diameter of the sphere.

749. Corollary.—If a perpendicular be let fall from the center of a sphere on the plane of a small circle, the foot of the perpendicular is the center of the circle; and conversely, the axis of any circle is a diameter of the sphere.

The two points where the axis of a circle pierces the spherical surface, are the *poles* of the circle. Thus, N and S are the poles of both the sections in the last diagram.

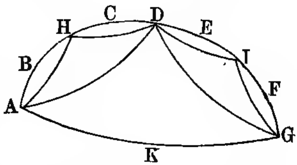
750. Corollary.—Circles whose planes are parallel to each other have the same axis and the same poles.

ARC OF A GREAT CIRCLE.

751. Theorem.—*The shortest line which can extend from one point to another along the surface of a sphere, is the arc of a great circle, passing through the two points.*

Only one great circle can pass through two given points on the surface of a sphere; for these two points and the center determine the position of the plane of the circle.

Let ABCDEFG be any curve whatever on the surface of a sphere from G to A. Let AKG be the arc of a great circle joining these points, and also AD and DG arcs of great circles joining those points with the point D of the given curve.



Then the sum of AD and DG is greater than AKG.

For the planes of these arcs form a triedral whose vertex is at the center of the sphere. These arcs have the same ratios to each other as the plane angles which compose this triedral, for the arcs are intercepted by the sides of the angles, and they have the same radius. But any one of these angles is less than the sum of the other two (586). Therefore, any one of the arcs is less than the sum of the other two.

Again, let AH and HD be arcs of great circles joining A and D with some point H of the given curve; also let DI and IG be arcs of great circles. In the same manner as above, it may be shown that AH and HD are greater than AD, and that the sum of DI and IG is greater than DG. Therefore, the sum of AH, HD, DI, and IG is still greater than AKG.

By continuing to take intermediate points and joining them to the preceding, a series of lines is formed, each greater than the preceding, and each approaching nearer to the given curve. Evidently, this approach can be made as nearly as we choose. Therefore, the curve is the limit of these lines, and partakes of their common character, in being greater than the arc of a great circle which joins its extremities.

752. Theorem.—*Every plane passing through the axis of a circle is perpendicular to the plane of that circle, and its section is a great circle.*

The first part of this theorem is a corollary of Article 556. The second part is proved by the fact that every axis passes through the center of a sphere (749).

753. Corollary.—The distances on the spherical surface from any points of a circumference to its pole, are the same. For the arcs of great circles which mark these distances are equal, since all their chords are equal oblique lines (529).

754. Corollary.—The distance of the pole of a great circle from any point of the circumference is a quadrant.

APPLICATIONS.

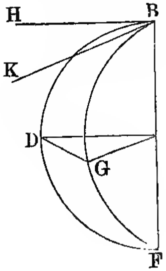
755. The student of geography will recognize the equator as a great circle of the earth, which is nearly a sphere. The parallels of latitude are small circles, all having the same poles as the equator. The meridians are great circles perpendicular to the equator.

The application of the principle of Article 751 to navigation has been one of the greatest reforms in that art. A vessel crossing the ocean from a port in a certain latitude to a port in the same latitude, should not sail along a parallel of latitude, for that is the arc of a small circle.

756. The curvature of the sphere in every direction, renders it impossible to construct an exact model with plane paper. But the student is advised to procure or make a globe, upon which he can draw the diagrams of all the figures. This is the more important on account of the difficulty of clearly representing these figures by diagrams on a plane surface.

SPHERICAL ANGLES.

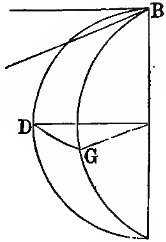
757. A SPHERICAL ANGLE is the difference in the directions of two arcs of great circles at their point of meeting. To obtain a more exact idea of this angle, notice that the direction of an arc at a given point is the same as the direction of a straight line tangent to the arc at that point. Thus, the direction of the arc BDF at the point B, is the same as the direction of the tangent BH.



758. Corollary.—A spherical angle is the same as the plane angle formed by lines tangent to the given arcs at their point of meeting. Thus, the spherical angle DBG is the same as the plane angle HBK, the lines HB and BK being severally tangent to the arcs BD and BG.

759. Corollary.—A spherical angle is the same as the diedral angle formed by the planes of the two arcs. For, since the intersection BF of the planes of the arcs is a diameter (748), the tangents HB and KB are both perpendicular to it, and their angle measures the diedral.

760. Corollary.—A spherical angle is measured by the arc of a circle included between the sides of the angle, the pole of the arc being at the vertex.



Thus, if DG is an arc of a great circle whose pole is at B, then the spherical angle DBG is measured by the arc DG.

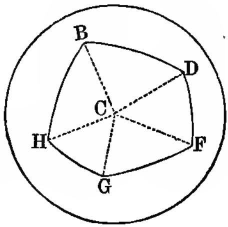
761. A Lune is that portion of the surface of a sphere included between two halves of great circles.

That portion of the sphere included between the two planes is called a *spherical wedge.* Hence, two great circles divide the surface into four lunes, and the sphere into four wedges.

SPHERICAL POLYGONS.

762. A Spherical Polygon is that portion of the surface of a sphere included between three or more arcs of great circles.

Let C be the center of a sphere, and also the vertex of a convex polyedral. Then, the planes of the faces of this polyedral will cut the surface of the sphere in arcs of great circles, which form the polygon BDFGH. We say *convex*, for only those polygons which have all the angles convex are considered among spherical polygons. Conversely, if a spherical polygon have the planes of its several sides produced, they form a polyedral whose vertex is at the center of the sphere.



The angles of the polygon are the same as the diedral angles of the polyedral (759).

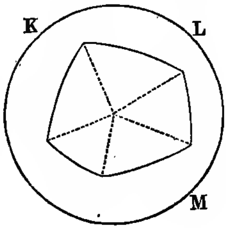
763. Theorem.—*The sum of all the sides of a spherical polygon is less than a circumference of a great circle.*

The arcs which form the sides of the polygon measure the angles which form the faces of the corresponding polyedral, for all the arcs have the same radius.

But the sum of all the faces of the polyedral being less than four right angles, the sum of the sides must be less than a circumference.

764. Theorem.—*A spherical polygon is always within the surface of a hemisphere.*

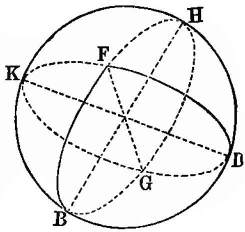
For a plane may pass through the vertex of the corresponding polyedral, having all of the polyedral on one side of it (609). The section formed by this plane produced is a great circle, as KLM. But since the polyedral is on one side of this plane, the corresponding polygon must be contained within the surface on one side of it.



765. That portion of a sphere which is included between a spherical polygon and its corresponding polyedral is called a *spherical pyramid*, the polygon being its base.

SPHERICAL TRIANGLES.

766. If the three planes which form a triedral at the center of a sphere be produced, they divide the sphere into eight parts or spherical pyramids, each having its triedral at the center, and its spherical triangle at the surface. Thus, for every spherical triangle, there are seven others whose sides are respectively either equal or supplementary to those of the given triangle.



Of these seven spherical triangles, that which lies vertically opposite the given triangle, as GKH to FDB, has its sides respectively equal to the sides of the given triangle, but they are arranged in reverse order; for the corresponding triedrals are symmetrical. Such spherical triangles are called *symmetrical.*

767. Corollary.—If two spherical triangles are equal, their corresponding triedrals are also equal; and if two spherical triangles are symmetrical, their corresponding triedrals are symmetrical.

768. Corollary.—On the same sphere, or on equal spheres, equal triedrals at the center have equal corresponding spherical triangles; and symmetrical triedrals at the center have symmetrical corresponding spherical triangles.

769. Corollary.—The three sides and the three angles of a spherical triangle are respectively the measures of the three faces and the three diedrals of the triedral at the center.

770. Corollary.—Spherical triangles are isosceles, equilateral, rectangular, birectangular, and trirectangular, according to their triedrals.

771. Corollary.—The sum of the angles of a spherical triangle is greater than two, and less than six right angles (591).

772. Corollary.—An isosceles spherical triangle is equal to its symmetrical, and has equal angles opposite the equal sides (594).

773. Corollary.—The radius being the same, two spherical triangles are equal,

1st. When they have two sides and the included angle of the one respectively equal to those parts of the other, and similarly arranged;

2d. When they have one side and the adjacent angles of the one respectively equal to those parts of the other, and similarly arranged;

3d. When the three sides are respectively equal, and similarly arranged;

4th. When the three angles are respectively equal, and similarly arranged.

774. Corollary.—In each of the four cases just given, when the arrangement of the parts is reversed, the triangles are symmetrical.

POLAR TRIANGLES.

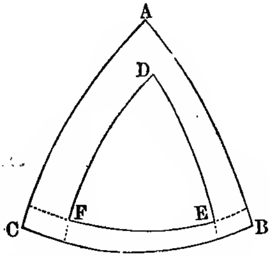
775. If at the vertex of a triedral, a perpendicular be erected to each face, these lines form the edges of a supplementary triedral (590). If the given vertex is at the center of a sphere, then there are two spherical triangles corresponding to these two triedrals, and they have all those relations which have been demonstrated concerning supplementary triedrals.

Since each edge of one triedral is perpendicular to the opposite face of the other, it follows that the vertex of each angle of one triangle is the pole of the opposite side of the other. Hence, such triangles are called *polar* triangles, though sometimes *supplementary.*

776. Theorem.—*If with the several vertices of a spherical triangle as poles, arcs of great circles be made, then a* *second triangle is formed whose vertices are also poles of the first.*

777. Theorem.—*Each angle of a spherical triangle is the supplement of the opposite side of its polar triangle.*

Let ABC be the given triangle, and EF, DF, and DE be arcs of great circles, whose poles are respectively A, B, and C. Then ABC and DEF are polar or supplementary triangles.



These two theorems are corollaries of Article 589, but they can be demonstrated by the student, with the aid of the above diagram, without reference to the triedrals.

778. The student will derive much assistance from drawing the diagrams on a globe. Draw the polar triangle of each of the following: a birectangular triangle, a trirectangular triangle, and a triangle with one side longer than a quadrant and the adjacent angles very acute.

INSCRIBED AND CIRCUMSCRIBED.

779. A sphere is said to be *inscribed* in a polyedron when the faces are tangent to the curved surface, in which case the polyedron is *circumscribed* about the sphere. A sphere is *circumscribed* about a polyedron when the vertices all lie in the curved surface, in which case the polyedron is *inscribed* in the sphere.

780. Problem.—*Any tetraedron may have a sphere inscribed in it; also, one circumscribed about it.*

For within any tetraedron, there is a point equally distant from all the faces (625), which may be the center of the inscribed sphere, the radius being the perpendicular distance from this center to either face. There is also a point equally distant from all the vertices of any tetraedron (623), which may be the center of the circumscribed sphere, the radius being the distance from this point to either vertex.

781. Corollary.—A spherical surface may be made to pass through any four points not in the same plane.

EXERCISES.

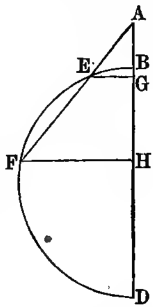
782.—1. In a spherical triangle, the greater side is opposite the greater angle; and conversely.

2. If a plane be tangent to a sphere, at a point on the circumference of a section made by a second plane, then the intersection of these planes is a tangent to that circumference.

3. When two spherical surfaces intersect each other, the line of intersection is a circumference of a circle; and the straight line which joins the centers of the spheres is the axis of that circle.

SPHERICAL AREAS.

783. Let AHF be a right angled triangle and BFD a semicircle, the hypotenuse AF being a secant, and the vertex F in the circumference. From E, the point where AF cuts the arc, let a perpendicular EG fall upon AD.



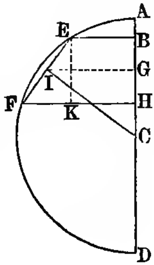
Suppose the whole of this figure to revolve about AD as an axis. The triangle AFH describes a cone, the trapezoid EGHF describes the frustum of a cone, and the semicircle describes a sphere.

The points E and F describe the circumferences of the bases of the frustum; and these circumferences lie in the surface of the sphere.

A frustum of a cone is said to be *inscribed* in a sphere, when the circumferences of its bases lie in the surface of the sphere.

784. Theorem.—*The area of the curved surface of an inscribed frustum of a cone, is equal to the product of the altitude of the frustum by the circumference of a circle whose radius is the perpendicular let fall from the center of the sphere upon the slant hight of the frustum.*

Let AEFD be the semicircle which describes the given sphere, and EBHF the trapezoid which describes the frustum. Let IC be the perpendicular let fall from the center of the sphere upon the slant hight EF.



Then the circumference of a circle of this radius would be π times twice CI, or 2πCI; and it is to be proved that the area of the curved surface of the frustum is equal to the product of BH by 2πCI.

The chord EF is bisected at the point I (187). From this point, let a perpendicular IG fall upon the axis AD. The point I in its revolution describes the circumference of the section midway between the two bases of the frustum. GI is the radius of this circumference, which is therefore 2πGI. The area of the curved surface of the frustum is equal to the product of the slant hight by this circumference (727); that is, EF by 2πGI.

Now from E, let fall the perpendicular EK upon FH. The triangles EFK and IGC, having their sides respectively perpendicular to each other, are similar. Therefore, EF : EK : : CI : GI. Substituting for the second term

EK its equal BH, and for the second ratio its equimultiple $2\pi CI : 2\pi GI$, we have

$$EF : BH :: 2\pi CI : 2\pi GI.$$

By multiplying the means and the extremes,

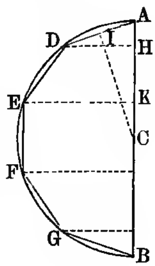
$$EF \times 2\pi IG = BH \times 2\pi IC.$$

But the first member of this equation has been shown to be equal to the area of the curved surface of the frustum. Therefore, the second is equal to the same area.

785. Corollary.—If the vertex of the cone were at the point A, the cone itself would be inscribed in the sphere; and there would be the same similarity of triangles, and the same reasoning as above. It may be shown that the curved surface of an inscribed cone is equal to the product of its altitude by the circumference of a circle whose radius is a perpendicular let fall from the center of the sphere upon the slant hight.

786. Theorem.—*The area of the surface of a sphere is equal to the product of the diameter by the circumference of a great circle.*

Let ADEFGB be the semicircle by which the sphere is described, having inscribed in it half of a regular polygon which may be supposed to revolve with it about the common diameter AB.

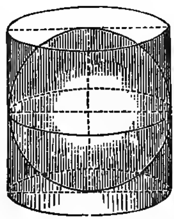


Then, the surface described by the side AD is equal to $2\pi CI$ by AH. The surface described by DE is equal to $2\pi CI$ by HK, for the perpendicular let fall upon DE is equal to CI; and so on. If one of the sides, as EF, is parallel to the axis, the measure is the same, for the surface is cylindrical. Adding these several equations together, we find that the entire surface described by the revolution of the regular polygon about its diameter, is equal to the product of the circumference whose radius is CI, by the diameter AB.

This being true as to the surface described by the perimeter of any regular polygon, it is therefore true of the surface described by the circumference of a circle. But this surface is that of a sphere, and the radius CI then becomes the radius of the sphere. Therefore, the area of the surface of a sphere is equal to the product of the diameter by the circumference of a great circle.

787. Corollary.—The area of the surface of a sphere is four times the area of a great circle. For the area of a circle is equal to the product of its circumference by one-fourth of the diameter.

788. Corollary.—The area of the surface of a sphere is equal to the area of the curved surface of a circumscribing cylinder; that is, a cylinder whose bases are tangent to the surface of the sphere.



AREAS OF ZONES.

789. A Zone is a part of the surface of a sphere included between two parallel planes. That portion of the sphere itself, so inclosed, is called a *segment*. The circular sections are the *bases* of the segment, and the distance between the parallel planes is the *altitude* of the zone or segment.

One of the parallel planes may be a tangent, in which case the segment has one base.

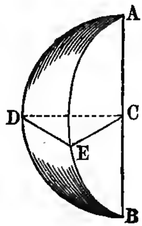
790. Theorem.—*The area of a zone is equal to the product of its altitude by the circumference of a great circle.*

This is a corollary of the last demonstration (786). The area of the zone described by the arc AD, is equal to the product of AH by the circumference whose radius is the radius of the sphere.

AREAS OF LUNES.

791. Theorem.—*The area of a lune is to the area of the whole spherical surface as the angle of the lune is to four right angles.*

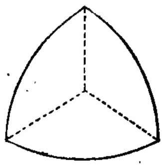
It has already been shown that the angle of the lune is measured by the arc of a great circle whose pole is at the vertex. Thus, if AB is the axis of the arc DE, then DE measures the angle DAE, which is equal to the angle DCE. But evidently the lune varies exactly with the angle DCE or DAE. This may be rigorously demonstrated in the same manner as the principle that angles at the center have the same ratio as their intercepted arcs.



Therefore, the area of the lune has the same ratio to the whole surface as its angle has to the whole of four right angles.

TRIRECTANGULAR TRIANGLE.

792. If the planes of two great circles are perpendicular to each other, they divide the surface into four equal lunes. If a third circle be perpendicular to these two, each of the four lunes is divided into two equal triangles, which have their angles all right angles and their sides all quadrants. Hence, this is sometimes called the *quadrantal* triangle.



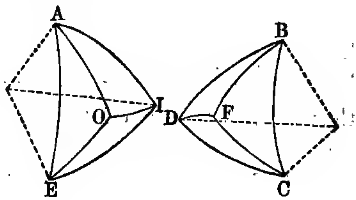
This triangle is the eighth part of the whole surface, as just shown. Its area, therefore, is one-half that of a great circle (787). Since the area of the circle is π times the square of the radius, the area of a trirectangular triangle may be expressed by $\frac{1}{2}\pi R^2$.

The area of the trirectangular triangle is frequently assumed as the unit of spherical areas.

AREAS OF SPHERICAL TRIANGLES.

793. Theorem.—*Two symmetrical spherical triangles are equivalent.*

Let the angle A be equal to B, E to C, and I to D. Then it is known that the other parts of the triangle are respectively equal, but not superposable; and it is to be proved that the triangles are equivalent.



Let a plane pass through the three points A, E, and I; also, one through B, C, and D. The sections thus made are small circles, which are equal; since the distances between the given points are equal chords, and circles described about equal triangles must be equal. Let O be that pole of the first circle which is on the same side of the sphere as the triangle, and F the corresponding pole of the second small circle. Let O be joined by arcs of great circles OA, OE, and OI, to the several vertices of the first triangle; and, in the same way, join FB, FC, and FD.

Now, the triangles AOI and BFD are isosceles, and mutually equilateral; for AO, IO, BF, and DF are equal arcs (753). Hence, these triangles are equal (772). For a similar reason, the triangles IOE and CFD are equal; also, the triangles AOE and BFC. Therefore, the triangles AEI and BCD, being composed of equal parts, are equivalent.

The pole of the small circle may be outside of the given triangle, in which case the demonstration would be by subtracting one of the isosceles triangles from the sum of the other two.

794. It has been shown that the sum of the angles of a spherical triangle is greater than the sum of the angles of a plane triangle (771). Since any spherical polygon can be divided into triangles in the same manner as a plane polygon, it follows that the sum of the angles of any spherical polygon is greater than the sum of the angles of a plane polygon of the same number of sides.

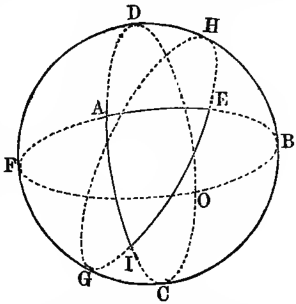
The difference between the sum of the angles of a spherical triangle, or other polygon, and the sum of the angles of a plane polygon of the same number of sides, is called the *spherical excess.*

795. Theorem.—*The area of a spherical triangle is equal to the area of a trirectangular triangle, multiplied by the ratio of the spherical excess of the given triangle to one right angle.*

That is, the area of the given triangle is to that of the trirectangular triangle, as the spherical excess of the given triangle is to one right angle.

Let AEI be any spherical triangle, and let DHBCGF be any great circle, on one side of which is the given triangle. Then, considering this circle as the plane of reference of the figure, produce the sides of the triangle AEI around the sphere.

Now, let the several angles of the given triangle be represented by a, e, and i; that is, taking a right angle for the unit, the angle EAI is equal to a right angles, etc. Then, the area of the lune AEBOCI is to the whole surface as a is to 4 (791). But if the trirectangular triangle, which is one-eighth of the spherical surface, be taken as the unit of area, then the area of this lune is $2a$. But the triangle BOC, which is a part of this lune, is equivalent to its opposite and symmetrical triangle DAF. Substituting this latter, the area of the two triangles ABC and DAF is $2a$ times the unit of area.



In the same way, show that the area of the two triangles IDH and IGC is $2i$, and that the area of the two triangles EFG and EHB is $2e$ times the unit of area. These equations may be written thus:

area (ABC + ADF) = $2a$ times the trirectangular triangle;

area (IDH + IGC) = $2i$ times the trirectangular triangle;

area (EFG + EHB) = $2e$ times the trirectangular triangle.

In adding these equations together, take notice that the triangles mentioned include the given triangle AEI three times, and all the rest of the surface of the hemisphere above the plane of reference once; also, that the area of this hemispherical surface is four times that of the trirectangular triangle. Then, by addition of the equations:

$$\text{area 4 trirect. tri.} + 2 \text{ area AEI} = 2(a + e + i) \text{ trir. tri.}$$

Transposing the first term, and dividing by 2.

$$\text{area AEI} = (a + e + i - 2) \text{ trir. tri.}$$

But $(a + e + i - 2)$ is the spherical excess of the given triangle, taking a right angle as a unit; that is, it is the ratio of the spherical excess of the given triangle to one right angle.

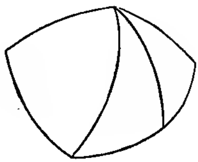
796. Corollary.—If the square of the radius be taken as the unit of area, then the area of any spherical triangle may be expressed (792),

$$\tfrac{1}{2}(a + e + i - 2)\pi R^2.$$

AREAS OF SPHERICAL POLYGONS.

797. Theorem.—*The area of any spherical polygon is equal to the area of the trirectangular triangle multiplied by the ratio of the spherical excess of the polygon to one right angle.*

For the spherical excess of the polygon is evidently the sum of the spherical excess of the triangles which compose it; and its area is the sum of their areas.



EXERCISES.

798.—1. What is the area of the earth's surface, supposing it to be in the shape of a sphere, with a diameter of 7912 miles?

2. Upon the same hypothesis, what portion of the whole surface is between the equator and the parallel of 30° north latitude?

3. Upon the same hypothesis, what portion of the whole surface is between two meridians which are ten degrees apart?

4. What is the area of a triangle described on a globe of 13 inches diameter, the angles being 100°, 45°, and 53°?

VOLUME OF THE SPHERE.

799. Theorem.—*The volume of any polyedron in which a sphere can be inscribed, is equal to one-third of the product of the entire surface of the polyedron by the radius of the inscribed sphere.*

For, if a plane pass through each edge of the polyedron, and extend to the center of the sphere, these planes will divide the polyedron into as many pyramids as the figure has faces. The faces of the polyedron are the bases of the pyramids.

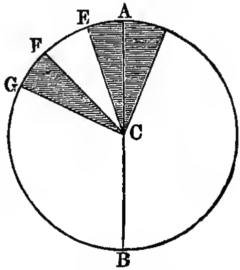
The altitude of each is the radius of the sphere, for the radius which extends to the point of tangency is perpendicular to the tangent plane (742). But the volume of each pyramid is one-third of its base by its altitude. Therefore, the volume of the whole polyedron is one-third the sum of the bases by the common altitude, or radius.

800. Theorem.—*The volume of a sphere is equal to one-third of the product of the surface by the radius.*

For, the surface of a sphere may be approached as nearly as we choose, by increasing the number of faces of the circumscribing polyedron, until it is evident that the sphere is the limit of the polyedrons in which it is inscribed. Then, this theorem becomes merely a corollary of the preceding.

801. Corollary.—The volume of a spherical pyramid, or of a spherical wedge, is equal to one-third of the product of its spherical surface by the radius.

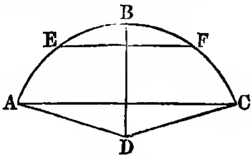
802. A spherical SECTOR is that portion of a sphere which is described by the revolution of a circular sector about a diameter of the circle. It may have two or three curved surfaces.



Thus, if AB is the axis, and the generating sector is AEC, the sector has one spherical and one conical surface; but if, with the same axis, the generating sector is FCG, then the sector has one spherical and two conical surfaces.

803. Corollary.—The volume of a spherical sector is equal to one-third of the product of its spherical surface by the radius.

804. The volume of a spherical segment of one base is found by subtracting the volume of a cone from that of a sector. For the sector ABCD is composed of the segment ABC and the cone ACD.



The volume of a spherical segment of two bases is the difference of the volumes of two segments each of one base. Thus the segment AEFC is equal to the segment ABC less EBF.

805. All spheres are similar, since they are generated by circles which are similar figures. Hence, we might at once infer that their surfaces, as well as their volumes, have the same ratios as in other similar solids. These principles may be demonstrated as follows:

806. Theorem. — *The areas of the surfaces of two spheres are to each other as the squares of their diameters; and their volumes are as the cubes of their diameters, or other homologous lines.*

For the superficial area of any sphere is equal to π times the diameter multiplied by the diameter (786); that is πD^2. But π is a certain or constant factor. Therefore, the areas vary as the squares of the diameters.

The volume is equal to the product of the surface by one-sixth of the diameter (800); that is, πD^2 by $\frac{1}{6}D$, or $\frac{1}{6}\pi D^3$. But $\frac{1}{6}\pi$ is a constant numeral. Therefore, the volumes vary as the cubes of the diameters.

USEFUL FORMULAS.

807. Represent the radius of a circle or a sphere, or that of the base of a cone or cylinder, by R; represent the diameter by D, the altitude by A, and the slant hight by H.

Circumference of a circle	$= \pi D = 2\pi R$,
Area of a circle	$= \frac{1}{4}\pi D^2 = \pi R^2$,
Curved surface of a cone	$= \frac{1}{2}\pi DH = \pi RH$,
Entire surface of a cone	$= \pi R(H+R)$,
Volume of a cone	$= \frac{1}{12}\pi D^2 A = \frac{1}{3}\pi R^2 A$,
Curved surface of a cylinder	$= \pi DA = 2\pi RA$,
Entire surface of a cylinder	$= 2\pi R(A+R)$,
Volume of a cylinder	$= \frac{1}{4}\pi D^2 A = \pi R^2 A$,
Surface of a sphere	$= \pi D^2 = 4\pi R^2$,
Volume of a sphere	$= \frac{1}{6}\pi D^3 = \frac{4}{3}\pi R^3$,

$$\pi = 3.1415926535.$$

EXERCISES.

808.—1. What is the locus of those points in space which are at the same distance from a given point?

2. What is the locus of those points in space which are at the same distance from a given straight line?

3. What is the locus of those points in space, such that the distance of each from a given straight line, has a constant ratio to its distance from a given point of that line?

EXERCISES FOR GENERAL REVIEW.

809.—1. Take some principle of general application, and state all its consequences which are found in the chapter under review; arranging as the first class those which are immediately deduced from the given principle; then, those which are derived from these, and so on.

2. Reversing the above operation, take some theorem in the latter part of a chapter, state all the principles upon which its proof immediately depends; then, all upon which these depend; and so on, back to the elements of the science.

3. Given the proportion, $a : b :: c : d$,

to show that $c-a : d-b :: a : b$;

also, that $a+c : a-c :: b+d : b-d$.

4. Form other proportions by combining the same terms.

5. What is the greatest number of points in which seven straight lines can cut each other, three of them being parallel; and what is the least number, all the lines being in one plane?

6. If two opposite sides of a parallelogram be bisected, straight lines from the points of bisection to the opposite vertices will trisect the diagonal.

7. In any triangle ABC, if BE and CF be perpendiculars to any line through A, and if D be the middle of BC, then DE is equal to DF.

8. If, from the vertex of the right angle of a triangle, there extend two lines, one bisecting the base, and the other perpen-

dicular to it, the angle of these two lines is equal to the difference of the two acute angles of the triangle.

9. In the base of a triangle, find the point from which lines extending to the sides, and parallel to them, will be equal.

10. To construct a square, having a given diagonal.

11. Two triangles having an angle in the one equal to an angle in the other, have their areas in the ratio of the products of the sides including the equal angles.

12. If, of the four triangles into which the diagonals divide a quadrilateral, two opposite ones are equivalent, the quadrilateral has two opposite sides parallel.

13. Two quadrilaterals are equivalent when their diagonals are respectively equal, and form equal angles.

14. Lines joining the middle points of the opposite sides of any quadrilateral, bisect each other.

15. Is there a point in every triangle, such that any straight line through it divides the triangle into equivalent parts?

16. To construct a parallelogram having the diagonals and one side given.

17. The diagonal and side of a square have no common measure, nor common multiple. Demonstrate this, without using the algebraic theory of radical numbers.

18. To construct a triangle when the three altitudes are given.

19. To construct a triangle, when the altitude, the line bisecting the vertical angle, and the line from the vertex to the middle of the base, are given.

20. If from the three vertices of any triangle, straight lines be extended to the points where the inscribed circle touches the sides, these lines cut each other in one point.

21. What is the area of the sector whose arc is 50°, and whose radius is 10 inches?

22. To construct a square equivalent to the sum, or to the difference of two given squares.

23. To divide a given straight line in the ratio of the areas of two given squares.

24. If all the sides of a polygon except one be given, its area will be greatest when the excepted side is made the diameter of a circle which circumscribes the polygon.

25. Find the locus of those points in a plane, such that the sum of the squares of the distances of each from two given points, shall be equivalent to the square of a given line.

26. Find the locus of those points in a plane, such that the difference of the squares of the distances of each from two given points, shall be equivalent to the square of a given line.

27. If the triangle DEF be inscribed in the triangle ABC, the circumferences of the circles circumscribed about the three triangles AEF, BFD, CDE, will pass through the same point.

28. The three points of meeting mentioned in Exercises 28, 29, and 30, Article 337, are in the same straight line.

29. If, on the sides of a given plane triangle, equilateral triangles be constructed, the triangle formed by joining the centers of these three triangles is also equilateral; and the lines joining their vertices to the opposite vertices of the given triangle are equal, and intersect in one point.

30. The feet of the three altitudes of a triangle and the centers of the three sides, all lie in one circumference. The circle thus described is known as "The Six Points Circle."

31. Four circles being described, each of which shall touch the three sides of a triangle, or those sides produced; if six lines be made, joining the centers of those circles, two and two, then the middle points of these six lines are in the circumference of the circle circumscribing the given triangle.

32. If two lines, one being in each of two intersecting planes, are parallel to each other, then both are parallel to the intersection of the planes.

33. If a line is perpendicular to one of two perpendicular planes, it is parallel to the other; and, conversely, if a line is parallel to one and perpendicular to another of two planes, then the planes are perpendicular to each other.

34. How may a pyramid be cut by a plane parallel to the base, so as to make the area or the volume of the part cut off have a given ratio to the area or the volume of the whole pyramid?

35. Any regular polyedron may have a sphere inscribed in it; also, one circumscribed about it.

36. In any polyedron, the sum of the number of vertices and the number of faces exceeds by two the number of edges.

37. How many spheres can be made tangent to three given planes?

38. Apply to spheres the principle of Article 331; also, of Article 191, substituting circles for chords.

39. Discuss the possible relative positions of two spheres.

40. What is the locus of those points in space, such that the sum of the squares of the distances of each from two given points, is equivalent to a given square?

41. What is the locus of those points in space, such that the difference of the squares of the distances of each from two given points, is equivalent to a given square?

42. A frustum of a pyramid is equivalent to the sum of three pyramids all having the same altitude as the frustum, and having for their bases the lower base of the frustum, the upper base, and a mean proportional between them.

43. The surface of a sphere can be completely covered with the surfaces either of 4, or of 8, or of 20 equilateral spherical triangles.

44. The volume of a cone is equal to the product of its whole surface by one-third the radius of the inscribed sphere.

45. If, about a sphere, a cylinder be circumscribed, also a cone whose slant hight is equal to the diameter of its base, then the area and volume of the sphere are two-thirds of the area and volume of the cylinder; and the area and volume of the cylinder are two-thirds of the area and volume of the cone.